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THE RATIONAL CUBOID AND A QUARTIC SURFACE

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1. The problem of solving in integers the system of Diophantine equations

$$(1)\qquad \begin{aligned} X^2 + Y^2 &= R^2 \\ Y^2 + Z^2 &= S^2 \\ Z^2 + X^2 &= T^2 \end{aligned}$$

has attracted much historical interest, see for example Dickson [**4**; Chapter XIX, references 1-29]. The numerical solution $(X, Y, Z) = (44, 117, 240)$ was observed as early as 1719, and Euler in 1772 provided the parametric solution

$$(2)\qquad \begin{aligned} X &= 8\lambda(\lambda-1)(\lambda+1)(\lambda^2+1) \\ Y &= (\lambda-1)(\lambda+1)(\lambda^2-4\lambda+1)(\lambda^2+4\lambda+1) \\ Z &= 2\lambda(\lambda^2-3)(3\lambda^2-1) \end{aligned}$$

although this was apparently discovered by Sanderson in 1740. Kraitchik [**6,7**] discusses the problem extensively and brings together many ad hoc methods for producing further parametric solutions. Of course the system (1) corresponds to a rectangular parallelepiped of which the edges and face diagonals are all integral. The further requirement that the cuboid diagonal be integral is given by the equation

$$(3)\qquad X^2 + Y^2 + Z^2 = \text{square},$$

and a great deal of effort has been spent in trying to decide the solvability or otherwise of equations (1) and (3). See Lal and Blundon [**8**], Leech [**9**] and Korec [**5**].

The approach here is to consider (1) geometrically, as the intersection V of three quadrics in five-dimensional projective space. There is a birational map to a quartic surface W, and it is this latter surface that we study. It possesses four isolated double points, and contains several pencils of elliptic curves. All the straight lines and conics

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