ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 14, Number 4, Fall 1984

CONSTRUCTING REAL PRIME DIVISORS USING NASH ARCS

ROBERT ROBSON

Dedicated to the memory of Gus Efroymson

Let $A = R[x_1, \ldots, x_n]$ be the affine coordinate ring of a variety V defined over the real closed field R. We denote the closed real points of V by $X \subset R^n$ and the simple points of X by $X_0 \subset X$. A geometric preorder P on the function field $K = R(x_1, \ldots, x_n)$ is a preorder corresponding to an (open) semialgebraic subset of X_0—in other words, there is an open semialgebraic set $U \subset X_0$ such that $f \in A \cap P$ precisely if $f \geqq 0$ on U.

Fix a geometric order P on K. If $B \subset K$ is any subring and $I \subset B$ is an ideal, we say that I is convex if $f \in I$ whenever $0 \leqq f \leqq g$ and $g \in I$. Here "$f \leqq g$" means $g - f \in P$. A valuation ring $(B, \mathfrak{m}) \subset K$ is said to be a real prime divisor if there is a domain $C \subset K$ of finite type over R and a minimal convex prime $\mathfrak{q} \subset C$ such that B is the localization $C_{(\mathfrak{q})}$. The theorem motivating this work is the following.

THEOREM *Let $\mathfrak{p} \subset A$ be a convex prime. Then there is a real prime divisor $(B, \mathfrak{m}) \subset K$ with $\mathfrak{m} \cap A = \mathfrak{p}$.*

Set $r = \text{tr.deg.}\ _R K$. In order to prove this theorem we construct $(r-1)$ functions $\xi_1, \ldots, \xi_{r-1} \in K$ and a total order $Q \subset K$ containing:

(A) P,

(B) $h^2(\xi_1, \ldots, \xi_{r-1}) - C_h^2$ for every non-zero polynomial $h \in R[T_1, \ldots, T_r]$ (pure polynomial ring) and some constants $C_h \in A \sim \mathfrak{p}$ depending on h, and

(C) $g^2 - f^2 h^2(\xi_1, \ldots, \xi_{r-1})$ for every $h \in R[T_1, \ldots, T_{r-1}]$, $g \in A \sim \mathfrak{p}$, and $f \in \mathfrak{p}$.

Once we know that such an order exists, it is a routine matter to show that the convex hull of the ring $A_{(\mathfrak{p})}[\xi_1, \ldots, \xi_{r-1}] \subset K$ in the order Q is our desired real prime divisor. Thus the hard part is defining $\xi_1, \ldots, \xi_{r-1}$ and proving the existence of Q.

Once the ξ_i are defined, Q exists providing that given any finite collection of inequalities from (A), (B), and (C) we may find a point $p \in U$ at which all the inequalities are fulfilled. Our definition of the ξ_i uses