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OPERATOR ALGEBRAS RELATED TO MEASURE PRESERVING TRANSFORMATIONS OF FINITE ORDER

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1. Introduction. The study of both multiplication and composition operators has a well established and productive history. In both cases questions of norms and spectra are, in varying degrees of complexity answered. This paper is concerned with the study of operators of the form $Tf = \sum_{i=0}^{N} a_i f \circ \tau^i$, acting on f in $L^2(X, \sum, m)$ where each a_i is a measurable function and τ is a measure preserving transformation on X. Special attention is paid to the case where $\tau^N(x) = x$ a.e. for some positive integer N, and τ is invertible. In this case we characterize the spectrum of T. The set of all such operators is shown to be a von Neumann algebra, and each such operator is shown to have a representation with the coefficient functions $a_0, \ldots, a_N$ in $L^\infty(X)$. The question of uniqueness of representation is answered. Finally a technique is developed enabling one to exhibit the coefficient functions concretely in terms of the operator itself.

2. Preliminaries and notation. Let $(X, \sum, m)$ be a complete finite measure space. For each set Y in $\sum$, l_Y represents both the characteristic function of Y and the act of restricting a σ-algebra, measure, or function to Y. A $\sum$-measurable mapping τ from X onto X is said to be measure preserving if $m(\tau^{-1}(A)) = m(A)$ for each set A in $\sum$ (equivalently $dm \circ \tau^{-1}/dm = 1$ a.e. dm). Throughout this article we assume that τ is both measure preserving and invertible. For any integrr k, τ^k represents the k-fold composition of τ with itself, with the obvious interpretation if $k = 0$. All statements about equality, inclusion and disjointness are to be understood to hold up to a set of m-measure 0. If V is a vector space and k is a positive integer then $V^{(k)}$ is the k-fold direct sum of V with itself. In case V is a Hilbert space we endow $V^{(k)}$ with the inner product

$$(\langle v_i \rangle, \langle u_i \rangle) = \sum_{i=0}^{k-1} (v_i, u_i).$$

For H a Hilbert space $B(H)$ is the ring of all bounded linear operators

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