ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 11, Number 4, Fall 1981

# THREE DIMENSIONAL HYPERBOLIC SPACES

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**1. Introduction.** Euclidean 3-space is a useful local model of the physical world. For two millennia from Euclid to Saccheri, mathematicians tried to prove that Euclidean geometry was the only consistent geometry of space. The effort continued until early in the 19th century when Bolyai, Lobachevsky, and Gauss independently investigated hyperbolic geometry. Later Riemann recognized spherical geometry as another non-Euclidean geometry and developed Riemannian geometry. By 1900, the axiomatic method and the role of a model in geometry were reasonably well understood. This viewpoint was spread by Hilbert's *Foundations of Geometry* and played an important part in the development of Einstein's theory of relativity. (See **[2]**, **[4]**, and **[11]**.) A homogeneous and isotropic 3-space with a curvature or scale factor which varies with time is a model for the large-scale spatial universe.

The use of the hyperbolic plane to study Riemann surfaces and Fuchsian groups was initiated by Poincaré and remains an active research area. Three dimensional spaces of constant negative curvature are less familiar. The recent work of W. P. Thurston **[16]** and others in this area is likely to have considerable impact in 3-manifold theory. In this paper we intend to give an account of some of the geometrical properties of negatively curved spaces. A complete Riemannian manifold of constant curvature is the quotient of a simply-connected manifold under the action of a discrete, fixed-point free, group of isometries. We will discuss several models for the covering space, the action of the isometries, volumes, some theorems about the quotients, and some examples.

**2. Hyperbolic models and volume.** The basic facts of riemannian geometry will be assumed. Let $\mathbf{R}^n$ denote Euclidean $n$-space and let $|x|^2 = x_1^2 + \cdots + x_n^2$. There are several useful models for hyperbolic $n$-space of curvature $-K < 0$. (See Wolf **[19]**.) The Poincaré disk model is $B^n(K) = \{x \in \mathbf{R}^n: |x|^2 < 1/K\}$ with the metric

$$\frac{4 \sum_{i=1}^{n} dx_i^2}{(1 - K|x|^2)^2}.$$

Received by the editors on May 24, 1979.