ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 9, Number 4, Fall 1979

ERGODIC THEOREMS FOR MIXING TRANSFORMATION GROUPS

JULIUS BLUM AND BENNETT EISENBERG

ABSTRACT. The notions of weak and strong mixing are extended to groups of transformations. Mixing transformations are characterized in terms of ergodic theorems which hold for those transformations.

0. **Introduction.** Let τ be a measure preserving transformation on a probability space (Ω, F, P) and let T denote the induced operator on L^2. We say T(or τ) is ergodic if the only functions left fixed by T are the constants. In this case the mean ergodic theorem says that $(1/N) \sum_{n=1}^{N} T^n f$ converges to $\int f\,dP$ in L^2 for f in L^2. Conversely, if $(1/N)\sum T^n f \to \int f\,dP$ for all f in L^2 then T is ergodic. (Convergence is in the L^2 sense throughout this paper.) Thus an ergodic transformation can be characterized as one whose "time averages" converge to the projection onto the constants, i.e., the "space average".

It also follows that T is ergodic if and only if

$$\frac{1}{N} \sum_{n=1}^{N} [(T^n f, f) - (\int f\,dP)^2] \to 0$$

for all f in L^2. This less intuitive propery of a transformation led to the definition of a strongly mixing transformation as one for which $(T^n f, f) \to (\int f\,dP)^2$ and of a weakly mixing transformation as one for which

$$\frac{1}{N} \sum_{n=1}^{N} |(T^n f, f) - (\int f\,dP)^2|^2 \to 0.$$

At first these concepts were not directly related to the ergodic problem if identifying time averages with space averages. But in 1960 Blum and Hanson [2] showed that a transformation is strongly mixing if and only if

$$\frac{1}{N} \sum_{k=1}^{N} T^{n_k} f \to \int f\,dP$$

for all subsequences n_k. In 1971 L. K. Jones [7] showed that a transformation is weakly mixing if and only if

Received by the editors on November 28, 1977.

AMS (MOS) Subject Classification number: Primary 28A65, 43A65, 43A25.

Key words and phrases: Mean ergodic theorem, mixing transformation group.