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THE GENERALIZED SPECTRUM OF SECOND ORDER ELLIPTIC SYSTEMS

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1. **Introduction.** Let $\Omega \subset R^n$ be a bounded domain with smooth boundary. Suppose that $L_{\alpha\beta}$, $\alpha, \beta = 1, 2, \cdots, m$ are second order elliptic operators without zero order terms which act on functions $u : \Omega \to \mathbf{C}^m$. The spectrum of the system

$$\sum_{\beta=1}^{m} L_{\alpha\beta} + \mu \sum_{\beta=1}^{m} c_{\alpha\beta} u^\beta = 0, \quad \alpha = 1, 2, \cdots, m$$

subject to appropriate homogeneous boundary conditions is known to consist of a discrete increasing set of numbers $\mu_1, \mu_2, \cdots, \mu_n, \cdots$.

In the case of a single equation with the Laplace operator as principal part and with homogeneous Dirichlet boundary conditions, a particularly simple method for obtaining a lower bound to the first eigenvalue μ_1 was obtained by Barta [1] who showed that

$$\mu_1 \geqq \inf_{x \in \Omega} \left(- \frac{\Delta \varphi}{\varphi} \right).$$

Here φ is an arbitrary C^2 function defined in Ω. This estimate is useful and of interest since the function φ is required to satisfy only a smoothness condition and not a boundary condition. This inequality was extended and generalized to general second order operators in [10]. There it is shown, for example, that μ_1, the first eigenvalue for the Laplace operator subject to zero boundary conditions satisfies the inequality

$$\mu_1 \geqq \inf_{x \in \Omega} (\operatorname{div} P - |P|^2)$$

where P is a vector field in Ω which is only required to satisfy a mild smoothness condition. The Barta inequality is recovered by setting $P_i = -\varphi_{x_i}/\varphi$ with φ an arbitrary C^2 function. Further extensions of these inequalities were obtained by Hersch [4]. Hooker [5] developed analogous results for second order equations with mixed boundary conditions and he also treated the eigenvalue problem for the biharmonic operator subject to a variety of boundary conditions.

Upper and lower bounds for the eigenvalues of second order operators have been obtained by a variety of methods. We mention the investigations of Fichera [3], Payne and Weinberger [9], Weinberger [12],

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