ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 7, Number 4, Fall 1977

ON THE PROBABILITY THAT AN INTEGER CHOSEN ACCORDING TO THE BINOMIAL DISTRIBUTION BE k-FREE

J. E. NYMANN AND W. J. LEAHEY

Introduction. Let s and t be integers chosen from among the first $n+1$ non-negative integers according to a binomial distribution with parameter p, $0 < p < 1$. Consider the probability that s and t be relatively prime. In [1] we showed that this probability tends to $6/\pi^2$, independent of p, as $n \to \infty$. Suppose now we choose a single integer s from the first $n+1$ non-negative integers according to a binomial distribution and ask what is the probability that s be square-free. In this paper we show that the techniques of [1] can also be used to show that this probability is $6/\pi^2$ in the limit. In fact we show something more, viz., that the probability that s be k-free, k any integer greater than 1, is $1/\zeta(k)$ where ζ denotes the Riemann zeta-function. (s is k-free if and only if s is not divisible by the k-th power of any prime.) In section 1 we deal with the case $k > 2$ and in section 2, with the case $k = 2$.

1. Let n be a non-negative integer and denote by N_n the set of integers $0, 1, 2, \cdots, n$. Let P_n be a probability distribution on N_n and let Q_k denote the set of non-negative k-free integers. Set $Q_k(n) = Q_k \cap N_n$. For any positive integer d, let $A_n(d) = \{j \in N_n : j \equiv 0 \pmod{d}\}$. We then have the following.

LEMMA 1. *Let P_n be any probability measure on N_n. Then for $n > 1$,*

$$P_n(Q_k(n)) = \sum_{1 \leqq d \leqq n^{1/k}} \mu(d)\{P_n(A_n(d^k)) - P_n(\{0\})\}.$$

PROOF. Let $p_1 < p_2 < \cdots < p_s$ be the primes less than or equal to $n^{1/k}$. Then, if $\tilde{Q}_k(n)$ denotes the complement of $Q_k(n)$ in N_n, we have

$$\tilde{Q}_k(n) = \bigcup_{i=1}^{s} A_n(p_i^{\,k}).$$

Therefore

Received by the editors on October 29, 1975, and in revised form on April 26, 1976.