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EVENTUALLY p-VALENT FUNCTIONS

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I. **Introduction.** In this paper we introduce the concept of eventual p-valence and investigate properties of eventually p-valent functions. Let $\mathcal{A}(D)$ denote the set of all functions $f(z)$ which are analytic in $|z| < 1$. We define a function $f(z)$ in $\mathcal{A}(D)$ to be *eventually p-valent* if there is a neighborhood $\mathcal{W}$ of infinity such that for all $w \in \mathcal{W}, f(z) = w$ has at most p roots in $|z| < 1$. Since eventual p-valence does not depend upon averaging properties, it is a natural concept for geometric function theory. Although the idea of eventual p-valence is hinted at in the literature, eventually p-valent functions have not been systematically investigated.

In Table 1 we summarize the necessary relations among functions which are eventually p-valent, areally mean p-valent, circumferentially p-valent or weakly p-valent. The proofs of some of these necessary relations are simplified by considering the set of all normalized locally univalent functions in $\mathcal{A}(D)$ with the real normed linear space structure introduced and developed in [11], [4]. We show that the concept of eventual p-valence is a meaningful linear invariant property while the notions of areal mean p-valence and circumferential mean p-valence are not meaningful linear invariant properties.

We define the *growth of an analytic function* in $\mathcal{A}(D)$ in terms of its maximum modulus $M(r,f)$ and show that *growth* f is a linear invariant concept. This allows us to form a new partitioning of the (universal) families $\mathcal{U}_\alpha$ of Pommerenke [20] and leads to the definition of the (universal) families $\mathcal{U}^\gamma$. If $f(z)$ is eventually p-valent, we show that $M(r,f) = O((1 - r)^{-2p})$.

We extend the Asymptotic Bieberbach Conjecture of G. Wing [22] to functions which are not even locally univalent. We conclude with four open questions suggested by the Extended Wing Theorem.

II. **Definitions and Preliminary Notions.** We let $\mathcal{J}$ denote the set of all Möbius transformations of $D = \{|z| < 1\}$ onto D and let $\mathcal{L}.\mathcal{S}.$ denote the set of all functions of the form $f(z) = z + \cdots$ which are analytic and locally univalent ($f'(z) \neq 0$) in D. If $\mathcal{M}$ is a family of functions in $\mathcal{L}.\mathcal{S}.$, we say that $\mathcal{M}$ is a *linear invariant family* [20, p. 112] if and only if for every $\phi(z)$ in $\mathcal{J}$, the function

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