

## THE DEVELOPMENT OF GENERAL DIFFERENTIAL AND GENERAL DIFFERENTIAL-BOUNDARY SYSTEMS

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I. **ABSTRACT.** The field of differential systems with general boundary conditions, differential-boundary systems is surveyed from 1900 to the present with special emphasis on the recent past. Results concerning dual systems, Green's matrices, eigenvalues and expansions, self-adjointness and applications (in particular to splines) are presented in such a way as to give a picture of the field as it has developed to the present. Finally several unsolved problems are listed.

II. **Introduction.** From various scattered places throughout the mathematical world certain problems, once thought of as side-lights to the field of ordinary boundary value problems, have recently coalesced into the small but vigorous new field of *general boundary value problems*. Consisting primarily of the study of ordinary differential systems under general boundary conditions and differential-boundary operators, the field has recently expanded to also include their applications to areas such as the calculus of variations, spline theory, hyperbolic dissipative systems and differential operators acting on subspaces of various  $L^2$  spaces. While the field moved extremely slowly at the start (in fact it lay dormant for one 20 year period), progress has been quite rapid during the last decade. With new applications being rapidly discovered, the field promises to be even more impressive in the future.

The purpose of this article is to give a general picture of the field as it developed, especially to describe in detail the interesting results of the past few years. At the beginning results are somewhat disjointed, due in part to the complexities involved. However as time passes, the field settles down into a well defined related group of problems. Before describing its evolution, however, let us consider several instances in which differential boundary problems have arisen, and what it is about them which interests the mathematician.

The 1952 Feller [22], while examining diffusion processes, encountered an interesting generalization of the Fokker-Planck equation:

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