ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 5, Number 3, Summer 1975

ON THE CONJUGATING REPRESENTATION OF A FINITE GROUP

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Very little is known about how the conjugating representation of a finite group decomposes into irreducible representations. In this note we investigate which sets of multiplicities are possible in such a decomposition. The analogous question for the regular representation is also long unsolved. Let $\nu_G = C \cdot 1_G + \gamma_G$ denote the conjugating representation of G (or its character). If c denotes the number of conjugacy classes of G, then the principal representation 1_G does not appear in the decomposition of γ_G. We show here that if G is not abelian (i.e., if $\gamma_G \neq 0$) then γ_G contains at least two inequivalent irreducible representations; moreover, if the group has trivial center and is not the symmetric group on three letters, then γ_G is not multiplicity-free; and if the group is simple, then the *g.c.d.* of the degrees of the irreducible constituents of γ_G is not divisible by the degree of any irreducible representation of G.

Recall that a primary representation is a direct sum of copies of a single irreducible representation.

LEMMA. *If γ_G is primary or multiplicity free then $\gamma_{G/Z(G)}$ must also be respectively primary or multiplicity free.*

PROOF. Let $C[G]$, $C[G/Z(G)]$ denote the complex group algebras of G and $G/Z(G)$ where $Z(G)$ denotes the center of G. Viewing these group algebras as left $C[G/Z(G)]$ modules with the action induced by conjugation, we see that $C[G/Z(G)]$ is a homomorphic image of $C[G]$ (under the mapping induced by $G \to G/Z(G)$) and so is in fact a direct summand of $C[G]$ by Maschke's Theorem. Hence if $C[G]$ is a direct sum of copies of the trivial module and copies of a single irreducible module, or a direct sum of copies of the trivial module and a multiplicity free module, then so is $C[G/Z(G)]$.

THEOREM 1. *γ_G is never primary unless $\gamma_G = 0$ and G is abelian.*

PROOF. Assume first that $Z(G) = 1$. If $\nu_G = c \cdot 1_G + a_\chi \chi$, $a_\chi \geqq 0$ then $1^*_{C(x_i)} = 1_G + m_i \chi$ for some $m_i \geqq 0$. Here $\{x_1, \cdots, x_c\}$ is a complete set of non-conjugate elements of G, $C(x_i)$ is the centralizer of x_i and $1^*_{C(x_i)}$ denotes the representation (or character) induced from the trivial representation of $C(x_i)$. Hence $h_i = [G : C(x_i)] = 1 + m_i \chi(1)$,

Received by the editors February 21, 1973.