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BOUNDS FOR RIEMANN-STIELTJES INTEGRALS

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ABSTRACT. Let f, g, h be real valued functions on a compact interval $[a, b]$, where h is of bounded variation with total variation V on $[a, b]$, and such that $\int_a^b f\,dg$ and $\int_a^b hf\,dg$ both exist. If $m = \inf\{h(x) : a \leqq x \leqq b\}$ it is shown that

$$\int_a^b hf\,dg \leqq m \int_a^b f\,dg + V \sup_{a \leqq \alpha < \beta \leqq b} \int_\alpha^\beta f\,dg,$$

$$\int_a^b hf\,dg \geqq m \int_a^b f\,dg + V \inf_{a \leqq \alpha < \beta \leqq b} \int_\alpha^\beta f\,dg.$$

Corresponding bounds hold for improper Riemann-Stieltjes integrals. The first of the inequalities above extends a result of R. Darst and H. Pollard, who dealt with the case $f(x) \equiv 1$, and g continuous on $[a, b]$.

In a recent paper [2], Darst and Pollard proved that *if h is real and of bounded variation on the interval $[a, b]$ and g is continuous there, then*

$$\int_a^b h\,dg \leqq (\inf h)[g(b) - g(a)] + V(h;\ [a, b])S_g(a, b), \tag{1}$$

where V is the total variation of h on $[a, b]$, and

$$S_g(a, b) = \sup_{a \leqq \alpha < \beta \leqq b} \int_\alpha^\beta dg. \tag{2}$$

Although it was not pointed out in [2], by replacing g in (1) by $(-g)$, one also obtains

$$\int_a^b h\,dg \geqq (\inf h)[g(b) - g(a)] + V(h;\ [a, b])s_g(a, b), \tag{1$'$}$$

where

$$s_g(a, b) = \inf_{a \leqq \alpha < \beta \leqq b} \int_\alpha^\beta dg. \tag{2$'$}$$

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