ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 4, Number 2, Spring 1974

CONTINUED FRACTION SOLUTIONS OF THE GENERAL RICCATI DIFFERENTIAL EQUATION

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The ideas presented here were developed in response to an observation made by H. S. Wall, whose memory as teacher and friend I cherish. In a class meeting of his course on continued fractions (in 1959–60, I think it was) he showed us how he had generated a continued fraction for a solution of the Riccati differential equation $y'(x) = x^2 + y^2(x)$. As an example for his undergraduate class in differential equations he had computed $y(x)$ as a ratio $u(x)/v(x)$ of power series satisfying $u'(x) = x^2 v(x)$, $v'(x) = -u(x)$, $u(0) = 0$, $v(0) = 1$. He had then applied repeated long division and obtained

$$y(x) = \frac{x^3}{3} - \frac{x^4}{7} - \frac{x^4}{11} - \frac{x^4}{15-} \cdots.$$

Recalling for us the similar continued fraction for $\tan x$, which satisfies the Riccati equation $y'(x) = 1 + y^2(x)$, he proposed the problem of generalizing these expansions. I took up the challenge and after some experimentation arrived at the following iterative scheme: if z is a complex number, and each of p_n, q_n, r_n, a_n, and b_n is a complex-valued function on the real interval I, then the recurrence relation $y_n = a_n{}^2/(a_n - b_n z - y_{n+1} z^2)$ converts the Riccati equation $y_n{}' = p_n + q_n y_n z + r_n y_n{}^2 z_n{}^2$ to the Riccati equation $y'_{n+1} = p_{n+1} + q_{n+1} y_{n+1} z + r_{n+1} y_{n+1}^2 z^2$ if

$$\begin{aligned} a_n{}' &= p_n,\ b_n{}' = q_n a_n, \\ p_{n+1} &= p_n(b_n/a_n)^2 - q_n b_n + r_n a_n{}^2, \\ q_{n+1} &= 2p_n(b_n/a_n{}^2) - q_n, \\ r_{n+1} &= p_n/a_n{}^2. \end{aligned} \tag{1}$$

Starting from $y_0 = y$, $p_0 = p$, $q_0 = q$, $r_0 = r$, this scheme produces, with minimal restrictions on p, q, and r, formal J-fraction solutions y of the general Riccati equation

$$y' = p + qyz + ry^2z^2, \tag{2}$$

namely

$$y = \frac{\alpha_0}{1 - \beta_0 z} - \frac{\alpha_1 z^2}{1 - \beta_1 z} - \frac{\alpha_2 z^2}{1 - \beta_2 z} - \cdots, \tag{3}$$

AMS (MOS) Subject Classifications (1970) Primary 34A45, Secondary 40A15.