JOURNAL OF COMMUTATIVE ALGEBRA
Volume 3, Number 2, Summer 2011

SOME PROPERTIES AND APPLICATIONS OF F-FINITE F-MODULES

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ABSTRACT. M. Hochster's work in [**7**] has shown that F-finite F-modules over regular local rings have finitely many F-submodules. In this paper we apply this theorem to prove that morphisms of F-finite F-modules have a particularly simple form and we also show that there exist finitely many submodules compatible with a given Frobenius near-splitting thus generalizing a similar result in [**1**] to the case where the base ring is not F-finite.

1. Introduction. The purpose of this paper is to describe several applications of finiteness properties of F-finite F-modules recently discovered by Hochster in [**7**] to the study of Frobenius maps on injective hulls, Frobenius near-splittings and to the nature of morphisms of F-finite F-modules.

Throughout this paper (R, m) shall denote a complete regular local ring of prime characteristic p. At the heart of everything in this paper is the Frobenius map $f : R \to R$ given by $f(r) = r^p$ for $r \in R$. We can use this Frobenius map to define a new R-module structure on R given by $r \cdot s = r^p s$; we denote this R-module $F_* R$. We can then use this to define the *Frobenius functor* from the category of R-modules to itself: given an R-module M we define $F_R(M)$ to be $F_* R \otimes_R M$ with R-module structure given by $r(s \otimes m) = rs \otimes m$ for $r, s \in R$ and $m \in M$. Henceforth we shall abbreviate F_R to F for the sake of readability.

Let $R[\Theta; f]$ be the skew polynomial ring which is the free R-module $\oplus_{i=0}^{\infty} R\Theta^i$ with multiplication $\Theta r = r^p \Theta$ for all $r \in R$. As in [**8**], $\mathcal{C}$ shall denote the category of $R[\Theta; f]$-modules which are Artinian as R-modules. For any two such modules M, N, we denote the morphisms between them in $\mathcal{C}$ with $\mathrm{Hom}_{R[\Theta;f]}(M, N)$; thus an element $g \in \mathrm{Hom}_{R[\Theta;f]}(M, N)$ is an R-linear map such that $g(\Theta a) = \Theta g(a)$ for

2010 AMS *Mathematics subject classification.* Primary 13A35, 13D45, 13P99.
The author gratefully acknowledges support from EPSRC grant EP/G060967/1.
Received by the editors on February 29, 2010, and in revised form on December 10, 2010.

DOI:10.1216/JCA-2011-3-2-225