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ON SUBSCHEMES OF 0-DIMENSIONAL SCHEMES WITH GIVEN GRADED BETTI NUMBERS

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ABSTRACT. We study the subschemes of a 0-dimensional scheme X for which either the Hilbert function or the graded Betti numbers are known. In the first case we find which kind of subscheme cannot stay in X and in the codimension 2 case what subschemes must be in X. In the case of graded Betti numbers we study the case of 2-codimensional partial intersection schemes or Artinian monomial ideals. More generally we give complete results for almost complete intersections and for other suitable Betti sequences.

Introduction. To understand the geometry of a 0-dimensional scheme X one should know which kind of subschemes it can contain. On this way we can see the Cayley-Bacharach property studied in [**3**] or the uniform position property stated in [**4**]. Of course, as much one knows about the 0-dimensional scheme X as much one can say about its subschemes. The first possible information regards the Hilbert functions of the subschemes. So one is interested in knowing the Hilbert functions of the subschemes of a scheme X having Hilbert function H. More precisely, one would like to know which Hilbert functions H' must necessarily live in X and which ones cannot stay in X. Then, if X is a 0-dimensional scheme of $\mathbf{P}^r$, H is a 0-dimensional O-sequence, $\psi = \Delta H$ one sets

$$\begin{aligned}
\mathcal{H}_X &= \{\varphi \mid \exists Y \subseteq X \text{ such that } \Delta H_Y = \varphi\}\\
\mathcal{H}_H^{(\mathrm{gen})} &= \{\varphi \mid \forall X \text{ with } \Delta H_X = \psi \ \exists Y \subseteq X \text{ with } \Delta H_Y = \varphi\}\\
\mathcal{H}_H &= \{\varphi \mid \exists X \text{ and } \exists Y \subseteq X \text{ with } \Delta H_X = \psi \text{ and } \Delta H_Y = \varphi\}.
\end{aligned}$$

Since $\mathcal{H}_H^{(\mathrm{gen})} \subseteq \mathcal{H}_X \subseteq \mathcal{H}_H$, where $H = H_X$, to have information about $\mathcal{H}_H^{(\mathrm{gen})}$ will mean to know which subschemes must necessarily

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