JOURNAL OF COMMUTATIVE ALGEBRA
Volume 3, Number 1, Spring 2011

COMULTIPLICATION MODULES OVER COMMUTATIVE RINGS

YOUSEF AL-SHANIAFI AND PATRICK F. SMITH

ABSTRACT. Let R be a commutative ring with identity. A unital R-module M is a comultiplication module provided for each submodule N of M there exists an ideal A of R such that N is the set of elements m in M such that $Am = 0$. It is proved that if M is a finitely generated comultiplication R-module with annihilator B in R then the ring R/B is semilocal and in certain cases M is quotient finite dimensional. Moreover, certain comultiplication modules satisfy the $AB5^*$-condition. If an R-module $X = \oplus_{i\in I} U_i$ is a direct sum of simple submodules U_i $(i \in I)$ and if P_i is the annihilator of U_i in R for each i in I then X is a comultiplication module if and only if $\cap_{j\neq i} P_j \nsubseteq P_i$ for all $i \in I$. A Noetherian comultiplication module is Artinian and a finitely generated Artinian module M is a comultiplication module if and only if the socle of M is a (finite) direct sum of pairwise non-isomorphic simple submodules. In case R is a Dedekind domain, an R-module M is a comultiplication module if and only if M is cocyclic or $M \cong (R/P_1^{k(1)}) \oplus \cdots \oplus (R/P_n^{k(n)})$ for some positive integers n, $k(i)$ $(1 \le i \le n)$ and distinct maximal ideals P_i $(1 \le i \le n)$ of R. For a general ring R a Noetherian R-module M is comultiplication if and only if the R_P-module M_P is comultiplication for every maximal ideal P of R, but it is shown that this is not true in general. It is shown that comultiplication modules and quasi-injective modules are related in certain circumstances.

1. Comultiplication modules. All rings are commutative with identity and all modules are unital. Let R be a ring, and let M be any R-module. Given submodules N and L of M we denote by $(N :_R L)$ the set of elements r in R such that $rL \subseteq N$. Note that $(N :_R L)$ is the annihilator in R of the R-module $(L + N)/N$ and is an ideal of R. In particular, if N is a submodule of M and $m \in M$ then $(N :_R Rm)$ will be denoted simply by $(N :_R m)$, so that $(N :_R m) = \{r \in R : rm \in N\}$. On the other hand, if N is again a submodule of M and A is an ideal of R then $(N :_M A)$ is the set of elements m in M such that $Am \subseteq N$

Received by the editors on October 28, 2009.
DOI:10.1216/JCA-2011-3-1-1