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HILBERT FUNCTIONS OF MULTIGRADED ALGEBRAS, MIXED MULTIPLICITIES OF IDEALS AND THEIR APPLICATIONS

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ABSTRACT. This paper is a survey on major results on Hilbert functions of multigraded algebras and mixed multiplicities of ideals, including their applications to the computation of Milnor numbers of complex analytic hypersurfaces with isolated singularity, multiplicities of blowup algebras and mixed volumes of polytopes.

1. Introduction. Let $R = \bigoplus_{n=0}^{\infty} R_n$ be a Noetherian graded algebra over a field $k = R_0$. Then R_n is a finite dimensional k-vector space. Consider the generating function

$$H(R,z) := \sum_{n=0}^{\infty} H_R(n) z^n$$

of the sequence $H_R(n) := \dim_k R_n$. By using *Hilbert's Syzygy theorem*, it can be proved that if $R = k[f_1, f_2, \dots, f_s]$ where $f_i \in R_{d_i}$ for $i = 1, 2, \dots, s$, then there exists a polynomial $h(z) \in \mathbf{Z}[z]$ such that

$$H(R,z) = \frac{h(z)}{(1-z^{d_1})(1-z^{d_2})\cdots(1-z^{d_s})}.$$

We say that R is standard if R is generated over R_0 by elements of degree 1. In this case the Hilbert function $H_R(n)$ is given by a polynomial $P_R(x) \in \mathbf{Q}[x]$ such that $H_R(n) = P_R(n)$ for all n large enough. Lasker [**37**] showed that the Krull dimension of R, denoted by $\dim R$, is $\deg P_R(x) + 1$. In the same paper, Lasker indicated that these results could be generalized to Hilbert functions of $\mathbf{N}^r$-graded algebras.

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