JOURNAL OF COMMUTATIVE ALGEBRA
Volume 1, Number 3, Fall 2009

ALGEBRAIC INTERPRETATION OF A THEOREM OF CLEMENTS AND LINDSTRÖM

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ABSTRACT. We study Hilbert functions of quotients of the truncated polynomial ring $k[x_1, \dots, x_n]/\left(x_1^{e_1+1}, x_2^{e_2+1}, \dots, x_n^{e_n+1}\right)$, where $e_1 \geq e_2 \geq \cdots \geq e_n \geq 1$ are integers. We use the work of Clements-Lindström to recover the well-known Macaulay's Theorem.

1. Introduction. Let $R = k[x_1, \dots, x_n]$, where k is a field, the x_i are indeterminates of degree 1, and I is a homogeneous ideal in R. Let $S = R/I$. Then $S = \oplus_{i\geq 0} S_i$ is a graded ring. The Hilbert function of S is defined by $H_S(i) = \dim_k S_i,\ i \geq 0$. By convention we take $H_S(i) = 0$ if $i < 0$. A sequence $\{c_i\}_{i\geq 0}$ such that $c_i = H_S(i),\ i \geq 0$ for some such S is called an O-sequence. In particular we have $c_0 = 1$. It is convenient to take $c_i = 0$ for $i < 0$. Macaulay characterized O-sequences combinatorially. Macaulay's characterization has been formulated by Stanley [**7**, Theorem 2.2 (i)⇔(iii)] in the form $c_0 = 1,\ c_i \geq 0$ for all $i \geq 0$, and $c_{i+1} \leq c_i^{<i>}$ for $i \geq 1$, where $c_i^{<i>}$ is defined in terms of binomial expansions. It is well known to commutative algebraists that the paper [**1**] of Clements and Lindström generalizes Macaulay's characterization of O-sequences to Hilbert functions of quotients of truncated polynomial rings of the form $k[x_1, \dots, x_n]/\left(x_1^{e_1+1}, x_2^{e_2+1}, \dots, x_n^{e_n+1}\right)$, where $e_1 \geq e_2 \geq \cdots \geq e_n \geq 1$ are integers. However [**1**] is written in a combinatorial language and it seems not to be as well understood how to interpret [**1**] algebraically. Greene and Kleitman give an exposition of the work of Clements and Lindström in [**3**] (also in a primarily combinatorial language). The purpose of this expository note is to describe our present understanding of how things work algebraically. In Section 2 we recall the results of Macaulay (as presented in [**7**]). In Section 3 we interpret [**1**] in terms of rev-lex-segments and order

2000 AMS *Mathematics subject classification.* Primary 13D40.
Keywords and phrases. Hilbert functions, rev-lex-segment ideals.
Received by the editors on November 12, 2007, and in revised form on May 4, 2008.

DOI:10.1216/JCA-2009-1-3-361 Copyright ©2009 Rocky Mountain Mathematics Consortium