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COUNTING THE NUMBER OF SOLUTIONS OF LINEAR CONGRUENCES

G. SBURLATI

ABSTRACT. We analyze some known formulas which concern counting the number of solutions of linear congruences and we find two important related numerical values which give an answer to interesting questions in elementary number theory related to distributions of sums modulo an integer. Two different ways to obtain good approximations of such values are discussed.

1. Introduction. Starting from known formulas giving the number of solutions of linear congruences with conditions on the greatest common divisor of each variable, in this paper a number of mathematical properties are derived which give an answer to questions like these: a finite set E of prime numbers being fixed, what are the integers favored as possible results of a sum having, for each prime p lying in E, a given number of addenda which are not multiples of p? If one also fixes a number $v \in \mathbf{N}$, how much can each single integer $m \in \mathbf{N}$ be favored or not favored if, for each p, in the sum there are at least v addenda not multiples of p? To answer the second question, we analyze from a qualitative and quantitative point of view two important values depending on v and related to linear congruences and, by proving two theorems and properties, we obtain their numerical expressions and two possible ways to calculate good approximations of them.

2. Some known results on linear congruences. We consider the problem of finding the elements $(x_1, x_2, \dots, x_k) \in \mathbf{Z}_r^k$ which satisfy the congruence equation

$$\sum_{j=1}^{k} h_j x_j \equiv a \pmod{r}, \tag{1}$$

and the constraining equalities

$$(x_j, r) = d_j; \quad j = 1, 2, \dots, k, \tag{2}$$

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