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ISOLS AND GENERALIZED BOOLEAN ALGEBRAS

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ABSTRACT. Let $\Gamma = \langle C, +, \cdot \rangle$ be a finite, hence atomic Boolean algebra. Then Γ is isomorphic to $\langle Q, \cup, \cap \rangle$, where Q is the family of all (finite) subsets of a (finite) set ν, namely the set of all atoms of Γ. Moreover, if ν has cardinality n, the Boolean algebra Γ is determined up to isomorphism by its order, i.e., 2^n, or equivalently by the number n. We shall extend this theorem to atomic generalized Boolean algebras $\Gamma = \langle C, +, \cdot \rangle$ in which the set C is isolated rather than finite. We have to impose some recursivity conditions on Γ which hold trivially, if Γ is finite. If these conditions are satisfied, Γ is effectively isomorphic to $\langle Q, \cup, \cap \rangle$, where Q is the family of all finite subsets of an isolated set ν, namely the set of all atoms of Γ. Moreover, if ν has RET (recursive equivalence type) N, the system Γ is determined up to effective isomorphism by its order, i.e., 2^N, or equivalently by the RET N. This result is of some interest, since the role played in ordinary arithmetic by the family of all (finite) subsets of some finite set ν is played in isolic arithmetic by the family of all finite subsets of some isolated set ν.

1. Algebraic preliminaries. Let $\Delta = \langle D, +, \cdot \rangle$ be a distributive lattice. For $u, v \in D$ we often abbreviate "$u \cdot v$" to "uv." The distributive lattice Δ has a zero-element if there is an element $0 \in D$ such that $x + 0 = x$ for all $x \in D$. Similarly, Δ has a one-element if there is an element $1 \in D$ such that $x \cdot 1 = x$ for all $x \in D$. If $p, q \in D$ we define $p \leq q$ as $pq = p$ or equivalently as $p + q = q$. For $a, b \in D$ we write $[a, b]$ for $\{x \in D \mid a \leq x \leq b\}$. A subset S of D is an *interval* of Δ if there are elements $a, b \in D$ such that $a \leq b$ and $S = [a, b]$. Note that, for $a, b, p, q \in D$,

$$(a \leq p \leq b \text{ and } a \leq q \leq b) \Rightarrow (a \leq p + q \leq b \text{ and } a \leq pq \leq b),$$

i.e., that $[a, b]$ is closed under $+$ and $\cdot$. Thus, by restricting the operations $+$ and $\cdot$ of Δ to the interval $[a, b]$ we obtain a distributive lattice with a as zero-element and b as one-element. This is called the lattice *induced by* Δ in $[a, b]$.

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