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ALGEBRA IS EVERYWHERE

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ABSTRACT. To various degrees, the invertibility or singularity of an operator between two different spaces can be reduced to that of a normed algebra element.

If an n-tuple $a \in A^n$ of normed algebra elements can be represented as a bounded linear operator $\operatorname{row}(L_a) : A^n \to A$ between normed spaces, and also as a bounded linear operator $\operatorname{col}(L_a) : A \to A^n$, then it is only fair that we should try to represent a bounded linear operator $T : X \to Y$ between different normed spaces by a system of normed algebra elements. In this note we see how various degrees of "invertibility" and "non-singularity" for $T \in \mathrm{BL}\,(X, Y)$ can be expressed in terms of the same thing for a related single element of the normed algebra $\mathrm{BL}\,(X \times Y, X \times Y)$ of operators on the cartesian product space, which we shall write in the form of column vectors:

$$\mathrm{BL}\left(\begin{pmatrix} X \\ Y \end{pmatrix}, \begin{pmatrix} X \\ Y \end{pmatrix}\right) = \begin{pmatrix} \mathrm{BL}\,(X, X) & \mathrm{BL}\,(Y, X) \\ \mathrm{BL}\,(X, Y) & \mathrm{BL}\,(Y, Y) \end{pmatrix}. \tag{0.1}$$

We begin by looking at "generalized inverses": we say [**3, 4**] that $T \in \mathrm{BL}\,(X, Y)$ is *regular*, or *relatively Fredholm*, if there is $T' \in \mathrm{BL}\,(Y, X)$ for which

$$T = TT'T, \tag{0.2}$$

and that $T \in \mathrm{BL}\,(X, Y)$ is *decomposably regular*, or *relatively Weyl*, if (0.2) can be arranged with invertible T'. Specializing to the case $Y = X$ and then generalizing, we shall say that an element $a \in A$ of a normed algebra A, or more generally an additive category A, is "regular" if

$$a \in aAa, \tag{0.3}$$

and "decomposably regular" if

$$a \in aA^{-1}a, \tag{0.4}$$