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PROBLEMS ABOUT REFLEXIVE ALGEBRAS

KENNETH R. DAVIDSON

The content of this paper was the subject of a talk given at the 1987 GPOTS meeting in Lawrence, Kansas. It was our purpose to try to survey some of the most important problems extant regarding reflexive algebras with a commutative invariant subspace lattice. In most cases, the motivation for these problems is based on a nice theory for nest algebras. However, the desired generalizations rarely turn out to be trivial; and often are not true in full generality.

We begin with two problems about nest algebras. Then we will briefly describe commutative subspace lattices (CSL's) and give a few examples before considering the remaining eight problems. Recall that a nest $\mathcal{N}$ is a family of closed subspaces of a Hilbert space containing $\{0\}$ and $\mathcal{H}$ which is totally ordered by inclusion and is complete with respect to intersection and closed span. The corresponding nest algebra $\mathcal{T}(\mathcal{N})$ is the algebra of all operators leaving each element of $\mathcal{N}$ invariant. The simplest example is given by taking an orthonormal basis $\{e_n, n \geq 1\}$ and forming $P_n = \operatorname{span}\{e_k, k \leq n\}$ for $n \geq 0$. Then $\mathcal{P} = \{P_n, n \geq 0; \mathcal{H}\}$ is a nest. Even for this simplest of all nests there is an interesting open problem.

PROBLEM 1. Is $\mathcal{T}(\mathcal{P})^{-1}$ connected?

It is surprising that such a simple sounding problem should remain open in this context. It is conjectured by many that the invertibles in $\mathcal{T}(\mathcal{P})$ are not connected. One reason is based on an analogy with function theory. The algebra H^∞ of bounded analytic functions on the unit disc is a nonselfadjoint subalgebra of L^∞ which is analogous to $\mathcal{T}(\mathcal{P})$ in $\mathcal{B}(\mathcal{H})$ in a number of ways. The set of Toeplitz operators $\{T_h^* : h \in H^\infty\}$ is a weak* closed abelian subalgebra of $\mathcal{T}(\mathcal{P})$ equal to the intersection of $\mathcal{T}(\mathcal{P})$ with the set of all Toeplitz operators. But the invertibles in H^∞ are not connected. Indeed, as in any commutative Banach algebra, the connected component of the identity is the set of

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