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A NOTE ON ORTHOGONAL POLYNOMIALS

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Let $d\mu$ be a finite positive Borel measure on the interval $[0, 2\pi]$ such that its support is an infinite set. Then there is a unique system $\{s_n\}_{n=0}^{\infty}$ of polynomials orthonormal with respect to $d\mu$ on the unit circle, i.e., polynomials

$$s_n(z) := s_n(d\mu, z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0, \qquad a_n := a_n(d\mu) > 0$$

satisfying

$$(1) \qquad \frac{1}{2\pi} \int_0^{2\pi} s_m(z)\overline{s_n(z)}\, d\mu(\theta) = \delta_{mn}, \qquad z = e^{i\theta};\ m, n > 0,$$

where $\delta_{mn} = 1$ if $m = n$ and $\delta_{mn} = 0$ otherwise. The purpose of this note is to give a simple and elementary proof of the following identity (without using any recurrence relations).

$$(2) \qquad \begin{gathered} \int_0^{2\pi} z^k |s_n(z)|^{-2}\, d\theta = \int_0^{2\pi} z^k\, d\mu(\theta), \\ z = e^{i\theta}, \quad |k| \le n,\ n = 0, 1, 2, \dots . \end{gathered}$$

This identity plays a very important role in the study of the asymptotics of orthonormal polynomials (cf., e.g., [**4, 5**]). Other proofs of (2) can be found in [**2**, Theorem 5.2.2, p. 198, **1**, Lemma 2 or **3**, formula (1.20), p. 7].

Proof of (2). For simplicity, we write $d\mu$ for $d\mu(\theta)$ and z for $e^{i\theta}$.

By (1), we have

$$\int_0^{2\pi} s_n(z) z^{-k}\, d\mu = \frac{2\pi}{a_n}\delta_{nk}, \qquad k = 0, 1, 2, \dots, n,$$

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