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A NATURAL EXTENSION OF A NONSINGULAR ENDOMORPHISM OF A MEASURE SPACE

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0. Introduction. Let (M, Σ, m) be a measure space. An endomorphism of M is a surjective map $S : M \to M$ such that $S^{-1}\Sigma \subseteq \Sigma$. An automorphism is a bijective map S such that S and S^{-1} are endomorphisms.

Since automorphisms are simple kinds of endomorphisms, establishing their properties can be easier than establishing those of general endomorphisms. In certain cases, an endomorphism S has a related automorphism T such that S and T have certain of the same properties. The sense in which T is related to S will be made more precise later. Such an automorphism T will be called a natural extension of S.

An endomorphism S is called measure-preserving if $m(S^{-1}E) = m(E)$ for every E in Σ. Rohlin [**6**, pp. 22–24] established that a measure-preserving endomorphism of a Lebesgue space has a natural extension. Implicit in his proof is the use of some kind of theorem on extension of measures. Cornfeld-Fomin-Sinai [**1**, pp. 239–240] proved Rohlin's result using the Daniell-Kolmogorov theorem, which needs the measure m to have a compact approximation property. Silva [**7**, pp. 8–11] extended Rohlin's result by constructing a natural extension of a nonsingular endomorphism of a standard Borel space. An endomorphism S is called nonsingular when $m(S^{-1}E) = 0$ if and only if $m(E) = 0$ for every E in Σ. Silva uses the skew-product construction to reduce to the measure-preserving case, and from that extension builds a natural extension for the nonsingular endomorphism. Lambert [**3**] claims to have a natural extension of an endomorphism of a general measure space. However, he assumes in addition to nonsingularity that $m(E) = 0$ implies $m(SE) = 0$. This condition is somewhat undesirable since it does not hold even in the measure-preserving case, as shown in the following example suggested by Choksi:

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