ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 28, Number 1, Spring 1998

# ON PROPERTIES OF $M$-IDEALS

JUAN CARLOS CABELLO AND EDUARDO NIETO

ABSTRACT. Given $r, s \in ]0,1]$, consider a Banach space $X$ which satisfies the following inequality

$$(*) \qquad \|f+g\| \geq r\|f\| + s\|g\|$$

for every $f$ in $X^*$ and $g$ in the annihilator of $X$ in $X^{***}$. It is well known that if $r = s = 1$, then $X$ is a WCG Asplund space, satisfying property $(u)$ of Pełczyński and property (A), i.e., every isometric isomorphism of $X^{**}$ is the bitranspose of an isometric isomorphism of $X$. The aim of this work is to show that, to have the above-mentioned properties, it is not necessary to suppose that $r = s = 1$. We prove, e.g., that $r + s > 1$ implies the Asplundness, $r = 1$ implies property $(u)$ (with $k_u(X) \leq 1/s$), and $s = 1$ implies $X$ is WCG satisfying property (A). Also many examples are given. For instance, a renormed James space $J$ satisfies $(*)$ for $s = 1$ and the renorming of $c_0$ by Johnson and Wolfe does not have property (A) and satisfies $(*)$ for $r = 1$.

**1. Introduction.** A Banach space $X$ is an *$M$-ideal* in its bidual, in short, $M$-ideal, if the equality $\|\varphi\| = \|\pi\varphi\| + \|\varphi - \pi\varphi\|$ holds for every $\varphi \in X^{***}$, where $\pi$ is the canonical projection of $X$, the natural projection from $X^{***}$ onto $X^*$. The class of $M$-ideals has been carefully investigated by Å. Lima, G. Godefroy and the "Berlin school", among others. As a consequence of these efforts, P. Harmand, D. Werner and W. Werner have published a recent monograph [**15**] which is considered the most systematic and complete study about this class. The spaces $c_0(I)$, $I$ any set, equipped with their canonical norm belong to this class, which also contains, e.g., certain spaces $\mathcal{K}(E,F)$ of compact operators between reflexive spaces, see, e.g., [**3, 14, 18** and **27**] or [**15**, Chapter VI]. $M$-ideals are known to enjoy many interesting isometric and isomorphic properties, e.g., they are weakly compactly generated (WCG) [**8**] and Asplund spaces [**20**], have properties $(u)$ (with constant one) and $(V)$ of Pełczyński [**11**] and [**12**], satisfy the

Received by the editors on February 28, 1996.
1991 *Mathematical Subject Classification*: 46B20.
Research partially supported by D.G.E.S., project No. PB96-1406.