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CONSTRUCTING COMPLETE PROJECTIVELY FLAT CONNECTIONS

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ABSTRACT. On any open subset U of the Euclidean space $\mathbf{R}^n$ there is complete torsion-free connection whose geodesics are reparameterizations of the intersections of the straight lines of $\mathbf{R}^n$ with U. For any positive integer m, there is a complete projectively flat torsion free connection on the two-dimensional torus such that for any point p there is another point q so that any broken geodesic from p to q has at least m breaks. This example is also homogeneous with respect to a transitive Lie group action.

1. Introduction. The purpose of this note is to tie up a couple of loose ends in the classical theory of linear connections. First, in [**6**, p. 395], Spivak raises the question of if, on a compact manifold with complete connection, any two points can be joined by a geodesic. The answer is "no" even when the connection is projectively flat and homogeneous:

Theorem 1. *Let T^2 be the two-dimensional torus. Then, for any positive integer m, there is a complete torsion free projectively flat connection, ∇, on T^2 such that for any point $p \in T^2$ there is a point $q \in T^2$ with the property that any broken ∇-geodesic between p and q has at least m breaks. Moreover if T^2 is viewed as a Lie group in the usual manner, this connection is invariant under translations by elements of T^2.*

Another natural question is: For a connected open subset, U, of the Euclidean space, $\mathbf{R}^n$, is the usual flat connection restricted to U projectively equivalent to complete torsion free connection on U? This is true and is a special case of a more general result about connections on incomplete Riemannian manifolds.

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