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THE FROBENIUS NUMBER AND a-INVARIANT

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ABSTRACT. We will give two different proofs for the fact that the Frobenius number of a numerical semigroup is the a-invariant of the semigroup algebra associated to it. These give rise to two different algorithms for computing the Frobenius number.

1. Introduction. Let $\mathcal{A} = \{w_1, \dots, w_n\}$ be a set of strictly positive integers and Q a subsemigroup of N generated by $\mathcal{A}$, i.e.,

$$Q = \langle \mathcal{A} \rangle = \mathrm{N}w_1 + \cdots + \mathrm{N}w_n.$$

We say that Q is numerical if the greatest common divisor of $\mathcal{A}$, $\gcd(\mathcal{A})$, is equal to 1, or equivalently $\mathrm{N} \setminus Q$ is a finite set [**9**, Exercise 10.2.4].

For the numerical semigroup Q the largest integer f^* not in Q is called the Frobenius number of Q, and the problem of finding this number is called the Frobenius problem. In other words, the problem is finding the largest integer f^* which cannot be written as a nonnegative integral combination of the w_i's. Thus the Frobenius number is concerned with a family of linear equations $\sum w_i x_i = f$, as f varies over all positive integers. The Frobenius problem has been examined by many authors ([**5, 6, 7**]).

Let k be a field, $k[\mathbf{x}] := k[x_1, \dots, x_n]$ the polynomial ring over k, $A := [w_1, \dots, w_n]$ an integer $1 \times n$-matrix whose entries generate the numerical semigroup Q, B an integer $n \times (n-1)$-matrix whose columns generate the lattice

$$\mathcal{L}_B := \mathrm{Ker}_{\mathrm{z}} A := \{u \in \mathrm{Z}^n : \ Au = 0\},$$

and $k[Q] \simeq k[t^{w_1}, \dots, t^{w_n}]$ the semigroup algebra associated to Q. For every $u \in \mathrm{Z}^n$ we define the body

$$P_u := \{v \in \mathrm{R}^{n-1} : \ Bv \leq u\}\cdot$$

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