ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 30, Number 1, Spring 2000

ALPERIN'S CONJECTURE AND THE SUBGROUP STRUCTURE OF A FINITE GROUP

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1. Introduction. In this mostly expository paper, we show how information about the subgroup structure of a finite group G can be used to determine certain numerical invariants of the group. Among the invariants we will consider are the number $k(G)$ of conjugacy classes of G and the closely related quantity $k_\pi(G)$, which is the number of classes of π-elements of G, where π is a set of prime numbers. For each prime p we will also consider the more subtle invariant $z_p(G)$, defined to be the number of ordinary irreducible characters of G that have p-defect zero. (Recall that these are the characters $\chi \in \operatorname{Irr}(G)$ such that the integer $|G|/\chi(1)$ is not divisible by p.) If p does not divide $|G|$, we see that every irreducible character has p-defect zero, and thus $z_p(G) = k(G)$ in this case. In general, however, it is much more difficult to determine $z_p(G)$ from the structure of G. The problem of doing this was proposed by Brauer [**1**] and solved by Robinson [**6**]. Unfortunately, Robinson's solution is fairly technical, and it does not seem to yield a direct algorithm for computing $z_p(G)$ from the subgroup structure of G.

The validity of our computation of class numbers is very easily established, but the application of our method to the determination of $z_p(G)$ relies on the unproved Alperin weight conjecture (AWC), which we shall explain in Section 3. (But the AWC is known to hold for p-solvable groups [**4**], and so our algorithm for determining $z_p(G)$ is definitely valid in that case.)

In fact, the AWC is one of the most significant outstanding problems in finite group representation theory, and it has been a focus of intense interest and activity by many researchers. It seems appropriate, therefore, that an account of this conjecture and of at least one of its potential applications should be made accessible to as wide an audience as possible. That is one of the goals of this paper, which we have

Received by the editors on April 24, 1998.
Research partially supported by a grant from the National Security Agency.