ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 31, Number 2, Summer 2001

FINITE GROUPS WITH EXACTLY TWO CONJUGACY CLASSES OF THE SAME ORDER

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ABSTRACT. We show that the only finite groups having a nilpotent derived subgroup and having exactly two conjugacy classes of the same order are $\mathbf{Z}_2$, D_{10}, the dihedral group of order 10, and A_4. As a corollary, the only supersolvable finite groups having exactly two conjugacy classes of the same order are $\mathbf{Z}_2$ and D_{10}.

1. Introduction. Investigation of the S_3-conjecture, that S_3 is the only finite group whose conjugacy classes all have different orders began with Markel [**7**] in 1973 and continued through a string of papers [**1**], [**3**], [**6**], [**9**] each examining a special case. Recently, the conjecture was confirmed in the class of finite solvable groups by Zhang [**10**] and, independently, by Knörr, Lempken and Thielcke [**5**].

Weakening the hypothesis on conjugacy class orders by allowing exactly two conjugacy classes to have the same order, one obtains a larger supply of examples. For instance, in the symmetric and alternating groups, where conjugacy classes are easy to compute [**8**, 11.1.1, 11.1.5], we find S_2, S_4, S_5, A_4, A_5 and A_7 each have exactly two conjugacy classes of the same order and those are the only such groups among the symmetric and alternating groups.

Here we begin a systematic study of which finite groups satisfy the weakened hypothesis where Markel began: with supersolvable groups. Specifically, we show that the only finite groups having a nilpotent derived subgroup and having exactly two conjugacy classes of the same order are $\mathbf{Z}_2$, D_{10} (the dihedral group of order 10) and A_4. As a corollary, the only supersolvable finite groups having exactly two conjugacy classes of the same order are $\mathbf{Z}_2$ and D_{10}.

The proof uses elementary techniques. However, the fact that a group satisfying our weakened hyypothesis is not necessarily a rational group

Portions of this work come from the first author's undergraduate honors thesis at Bucknell University.
Received by the editors on January 9, 1996, and in revised form on October 15, 1996.