ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 31, Number 2, Summer 2001

THE WORPITZKY-PRINGSHEIM THEOREM ON CONTINUED FRACTIONS

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ABSTRACT. We compare the classical convergence theorems of Worpitzky and Pringsheim in the theory of continued fractions. We give an extension of Worpitzky's theorem, and we also discuss inequalities concerning the limit point-limit circle dichotomy.

1. Introduction. Given complex numbers a_n and b_n, $n = 1, 2, \dots,$ where $a_n \neq 0$ for every n, the continued fraction

$$\mathbf{K}(a_n \mid b_n) = \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cdots}}} \tag{1.1}$$

converges to the value k if $T_n(0) \to k$ as $n \to \infty$, where

$$t_n(z) = a_n/(z + b_n), \quad T_n = t_1 \circ t_2 \circ \cdots \circ t_n. \tag{1.2}$$

Throughout this paper t_n and T_n will be defined by (1.2).

In 1865 Worpitzky proved that if $|a_n| \leq 1/4$ for all n, then $\mathbf{K}(a_n \mid 1)$ converges, and after rescaling the complex plane by a factor 2 we can express this as follows.

Worpitzky's theorem. *If $|a_n| \leq 1$ for all n, then $\mathbf{K}(a_n \mid 2)$ converges.*

Worpitzky's theorem was later generalized by Pringsheim who, in 1889, proved the following result, see [**4**, pp. 92–94], [**5**, pp. 30–35], [**6**, p. 58] and [**8**, p. 42].

AMS *Mathematics Subject Classification.* 30B70.
Received by the editors on October 20, 1999.