ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 40, Number 3, 2010

RANKS OF CROSS COMMUTATORS ON BACKWARD SHIFT INVARIANT SUBSPACES OVER THE BIDISK

KEI JI IZUCHI AND KOU HEI IZUCHI

ABSTRACT. On backward shift invariant subspaces M over the bidisk, it is proved that if M is Hilbert-Schmidt, then

$$\operatorname{rank}[R_z, R_w^*] - 1 \leq \operatorname{rank}[S_z, S_w^*] \leq \operatorname{rank}[R_z, R_w^*] + 1.$$

1. Introduction. Let Γ^2 be the two-dimensional unit torus. We write z, w for variables in $\Gamma^2 = \Gamma_z \times \Gamma_w$. Let $L^2 = L^2(\Gamma^2)$ be the space of square integrable functions on Γ^2 with the inner product

$$\langle f, g\rangle = \int_0^{2\pi}\int_0^{2\pi}(f\overline{g})(e^{is}, e^{it})\frac{dsdt}{(2\pi)^2}.$$

Let $H^2 = H^2(\Gamma^2)$ be the Hardy space over Γ^2, and let $H^2(\Gamma_z)$ and $H^2(\Gamma_w)$ be the Hardy spaces on the unit circle Γ with variables z and w, respectively. We think of $H^2(\Gamma_z)$ and $H^2(\Gamma_w)$ as closed subspaces of H^2. For a function ψ in $L^\infty(\Gamma^2)$, the Toeplitz operator T_ψ on H^2 is defined by $T_\psi f = P(\psi f)$, where P is the orthogonal projection from L^2 onto H^2. It is known that $T_\psi^* = T_{\overline{\psi}}$, and $T^*_{\varphi(z)}T_{\psi(w)} = T_{\psi(w)}T^*_{\varphi(z)}$ for every $\varphi(z), \psi(w) \in H^\infty(\Gamma)$. A nonzero closed subspace M of H^2 with $M \neq H^2$ is called invariant if $T_zM \subset M$ and $T_wM \subset M$. A function f in H^2 is called inner if $|f| = 1$ almost everywhere on Γ^2. A well-known theorem due to Beurling [**2**] says that an invariant subspace M of $H^2(\Gamma_z)$ has a form $M = q(z)H^2(\Gamma_z)$ for an inner function $q(z)$. In the two variables case, the structure of invariant subspaces of H^2 is extremely complicated, see [**3, 12**].

2010 AMS *Mathematics subject classification.* Primary 47A15, 32A35, Secondary 47B35.
Keywords and phrases. Backward shift invariant subspace, invariant subspace, Hardy space, cross commutator.
Received by the editors on December 13, 2007, and in revised form on February 14, 2008.
DOI:10.1216/RMJ-2010-40-3-929