ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 40, Number 3, 2010

CONVEXITY OF THE INTEGRAL ARITHMETIC MEAN OF A CONVEX FUNCTION

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ABSTRACT. In this paper it is proved that the integral arithmetic mean of a continuous function f is a convex function if and only if f is a convex function.

1. Introduction. For the convenience of the readers, we recall the main definitions as follows.

Definition 1. Let $D \subset \mathbf{R}^n$ be a convex set (if $n = 1$, then D is an interval). A function $f : D \to \mathbf{R}^n$ is called a convex function on D if

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}$$

for all $x, y \in D$.

Definition 2. Let I be an interval with nonempty interior. A function $F : I^n \to \mathbf{R}$ is called a Schur-convex function on I^n if

$$F(x_1, x_2, \ldots, x_n) \leq F(y_1, y_2, \ldots, y_n)$$

for any two n-tuples $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$ in I^n, such that $x \prec y$ holds, i.e.,

$$\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}, \quad k = 1, 2, \ldots, n-1$$

2010 AMS *Mathematics subject classification.* Primary 26D15.
Keywords and phrases. Convex function, Schur-convex function, arithmetic mean, positive semi-definite matrix.
The research of this paper was supported by the 973 Project of China under grant 2006CB708304, N.S. Foundation of China under grant 60850005 and N.S. Foundation of the Zhejiang Province under grant Y607128.
Received by the editors on July 17, 2007, and in revised form on February 12, 2008.
DOI:10.1216/RMJ-2010-40-3-1061