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THE Q-ANALOGUE OF STIRLING'S FORMULA

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ABSTRACT. F.H. Jackson defined a q-analogue of the factorial $n! = 1\cdot 2\cdot 3 \cdots n$ as $(n!)_q = 1\cdot(1+q)\cdot(1+q+q^2)\cdots(1+q+q^2+\cdots+q^{n-1})$, which becomes the ordinary factorial as $q \to 1$. He also defined the q-gamma function as

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x},\ 0 < q < 1,$$

and

$$\Gamma_q(x) = \frac{(q^{-1};q^{-1})_\infty}{(q^{-x};q^{-1})_\infty}(q-1)^{1-x} q^{\binom{x}{2}},\ q > 1,$$

where

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n).$$

It is known that if $q \to 1$, $\Gamma_q(x) \to \Gamma(x)$, where $\Gamma(x)$ is the ordinary gamma function. Clearly $\Gamma_q(n+1) = (n!)_q$, so that the q-gamma function does extend the q factorial to non integer values of n. We will obtain an asymptotic expansion of $\Gamma_q(z)$ as $|z| \to \infty$ in the right halfplane, which is uniform as $q \to 1$, and when $q \to 1$, the asymptotic expansion becomes Stirling's formula.

1. Introduction. In recent years many of the classical facts about the ordinary gamma function have been extended to the q-gamma function. See Askey [2], and [5], [6]. Using an identity of Euler,

$$\frac{1}{(x;q)_\infty} = \sum_{n=0}^{\infty}\frac{x^n}{(q;q)_n}, \tag{1.1}$$

$\Gamma_q(x)$ can be written as,

$$\Gamma_q(x) = (q;q)_\infty(1-q)^{1-x}\sum_{n=0}^{\infty}\frac{q^{nx}}{(q;q)_n},\ 0 < q < 1, \tag{1.2}$$

and

$$\Gamma_q(x) = (q^{-1};q^{-1})_\infty\, q^{\binom{x}{2}}(q-1)^{1-x}\sum_{n=0}^{\infty}\frac{q^{-nx}}{(q^{-1};q^{-1})_n},\ q > 1. \tag{1.3}$$

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