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# THE CONTROL MEASURE PROBLEM AND THE UNIVERSAL MEASURE TOPOLOGY

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In this paper we characterize submeasures with control measures and pathological submeasures in terms of the universal measure topology of Graves. The results give new interpretations of several problems equivalent to the control measure problem (Maharam submeasure problem). Our main tool is the complete Boolean algebra of projections in the universal measure space; the setting for our results is the complete lattice of Fréchet-Nikodým topologies on an algebra of sets.

**1. Preliminaries.** Let $\mathscr{B}$ be a Boolean algebra. A *submeasure* on $\mathscr{B}$ is a map $\lambda\colon \mathscr{B} \to [0, \infty)$ such that

(1) $\lambda(0) = 0$,

(2) $\lambda(A) \leqq \lambda(B)$ whenever $A \leqq B$ in $\mathscr{B}$,

(3) $\lambda(A \vee B) \leqq \lambda(A) + \lambda(B)$ for all $A$ and $B$ in $\mathscr{B}$.

A submeasure $\lambda$ is *exhaustive* if $\lambda(A_n) \to 0$ whenever $(A_n)$ is a disjoint sequence in $\mathscr{B}$ and *continuous* if $\lambda(A_n) \to 0$ whenever $(A_n)$ is a decreasing sequence in $\mathscr{B}$ and $\bigwedge A_n = 0$. Call $\lambda$ *strictly positive* when $\lambda(A) = 0$ if and only if $A = 0$.

Let $\mathscr{C}$ be the algebra of clopen subsets of the Stone space $X$ of $\mathscr{B}$. Then submeasures on $\mathscr{B}$ are in $1 - 1$ correspondence with submeasures on $\mathscr{C}$.

For submeasures $\lambda$ and $\mu$ on $\mathscr{C}$, say $\lambda$ is *$\mu$-continuous* if for every $\varepsilon > 0$ there is $\delta > 0$ such that $\lambda(A) < \varepsilon$ whenever $\mu(A) < \delta$. Call $\lambda$ and $\mu$ *mutually continuous* if $\lambda$ is $\mu$-continuous and $\mu$ is $\lambda$-continuous.

A *Fréchet-Nikodým topology* on $\mathscr{C}$ is a topology making the map $(A, B) \to A\Delta B$ from $\mathscr{C} \times \mathscr{C}$ (with the product topology) to $\mathscr{C}$ continuous and making the map $A \to A \cap B$ continuous at $\emptyset$ uniformly for $B$ in $\mathscr{C}$. A Fréchet-Nikodým topology on $\mathscr{C}$ makes $\mathscr{C}$ into a topological group in which intersection is uniformly continuous. Fréchet-Nikodým topologies were introduced and studied by Drewnowski [5], [6].

If $\lambda$ is a submeasure on $\mathscr{C}$, then we may define a semimetric $d_\lambda$ on $\mathscr{C}$ by $d_\lambda(A, B) = \lambda(A\Delta B)$, Of course, $d_\lambda$ is a metric if and only if $\lambda$ is strictly positive. The semimetric topology $G_\lambda$ is a Fréchet-Nikodým topology

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