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ON LIE GROUPS WITH MINIMAL GENERATING SETS OF ORDER EQUAL TO THEIR DIMENSION

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ABSTRACT. Let G be a connected Lie group with Lie algebra g, $\{X_1, \ldots, X_\ell\}$ a minimal generating set for g. The order of generation of G with respect to $\{X_1, \ldots, X_\ell\}$ is the smallest integer M such that every element of G can be written as a product of M elements taken from $\exp(tX_1), \ldots, \exp(tX_\ell)$. We find all G which admit minimal generating sets $\{X_1, \ldots, X_n\}$ with $n = \dim G$; for each such set we construct an algorithm for computing the order of generation of G.

I. Introduction. A connected Lie group G is generated by one-parameter subgroups $\exp(tX_1), \ldots, \exp(tX_\ell)$ if every element of G can be written as a finite product of elements chosen from these subgroups. In this case, define the order of generation of G to be the least positive integer M such that every element of G possesses such a representation of length at most M; if no such integer exists let the order of generation of G be infinity. The order of generation will, of course, depend upon the one-parameter subgroups. Computation of the order of generation of G for given $X_1, \ldots, X_\ell$ is analogous to finding the greatest wordlength needed to write each element of a finite group in terms of generators $g_1, \ldots, g_\ell$.

The subgroups $\exp(tX_1), \ldots, \exp(tX_\ell)$ generate G just in case $X_1, \ldots, X_\ell$ generate the Lie algebra g of G. Indeed the set of all finite products of elements from $\exp(tX_1), \ldots, \exp(tX_\ell)$ is an arcwise connected subgroup of G and so a Lie subgroup by Yamabe's theorem **[10]**; clearly the Lie algebra of this subgroup is the subalgebra of g generated by $X_1, \ldots, X_\ell$.

It is natural to restrict attention to minimal generating sets; from now on, then, suppose that no subset of $\{X_1, \ldots, X_\ell\}$ generates g. Call two generating sets $\{X_1, \ldots, X_\ell\}$ and $\{Y_1, \ldots, Y_\ell\}$ *equivalent* if it is possible to find an automorphism σ of G, a permutation τ of $\{1, \ldots, \ell\}$, and non-zero constants $\lambda_1, \ldots, \lambda_\ell$ such that $X_i = \lambda_i \sigma_*(Y_{\tau(i)})$. The order of generation of G depends only on the equivalence class of the generating set.

If $\{X_1, \ldots, X_\ell\}$ is a minimal generating set for G and $\dim G > 1, 2 \leqq \ell$

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