ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 11, Number 4, Fall 1981

ARITHMETIC PROPERTIES OF THE MÉNAGE POLYNOMIALS

HARLAN STEVENS

1. Introduction. The ménage polynomials $U_n(t)$ are defined for $n > 1$ by

$$U_n = U_n(t) = \sum_{k=0}^{n} u_{n,k} t^k,$$

where $u_{n,k}$ is the number of ways of seating n married couples at a circular table, men and women alternating, so that exactly k husbands sit next to their own wives. The numbers $u_{n,k}$ are to be thought of as 'reduced' in that the positions of the women are taken as fixed. A comprehensive account of the *problème des ménage* is given by Riordan and Kaplansky in [3].

Riordan [4] has shown that the ménage polynomials possess a rather simple periodic property. He proved, namely, that when $U_0 = 2$, $U_1 = 2t - 1$

$$U_{n+p^2} \equiv (t^{p^2} - 1)U_n \pmod{p} \tag{1.1}$$

for all $n \geqq 0$ and odd primes p. In this note we will show that the U_n actually satisfy a much wider class of congruences. It will be demonstrated in fact that if $m = cp^e$, then

$$\sum_{s=0}^{r} (-1)^s \binom{r}{s} (t-1)^{m(r-s)} U_{n+sm} \equiv 0 \pmod{p^{(e-1)r+r_1}} \tag{1.2}$$

for $n \geqq 0$ and where $r_1 = [r/2]$ is the greatest integer $\leqq r/2$. This last notation for r_1 will be maintained throughout. The congruence (1.2) reduces to (1.1) when $m = p^2$ and $r = 1$.

In [1] Carlitz also considered congruences like (1.2). His results, however, coincide with ours only for the cases $e = 1$ or $r \leqq 2$, but are otherwise weaker. Moreover, the method of the present paper is very direct and avoids much of the computation of both [1] and [4].

It is of interest to note here that the congruences represented by (1.2) are quite reminiscent of those satisfied by Hermite and Laguerre polynomials [2]. In spite of these similarities and the fact that they all obey difference equations of the second order, it is curious that the proofs in each case are apparently unrelated.

Received by the editors on October 1, 1979.