ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 11, Number 4, Fall 1981

CHOICE SETS AND MEASURABLE SETS

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Call two real numbers *equivalent* if their difference is rational. Call $S \subset R$ a *choice set* if S is a set of representatives of the equivalence classes of R. J. A. Andrews [1] observed that the set $\{\lambda S: S \in \mathscr{F}\}$ is dense in the unit interval [0, 1] where λ denotes Lebesgue outer measure and $\mathscr{F}$ denotes the family of all choice sets $\subset [0, 1]$. In this note we prove that in fact $\{\lambda S: S \in \mathscr{F}\} = (0, 1]$. More generally we prove the following theorem.

THEOREM 1. *There exists a set $E \subset R$ such that*

(i) *$\lambda(E \cap A) = \lambda(A)$ where A is any Lebesgue measurable set, and*

(ii) *$E \cap (r + E) = \emptyset$ where r is any nonzero rational number. Moreover, if I is any interval in R, and S is any extension of the set $E \cap I$ to a choice set $S \subset I$, then $\lambda S = \lambda I$.*

PROOF. Let Q be the field of rational numbers. Say that $x, y \in R\backslash Q$ are Q-equivalent if $y \in Qx + Q$. This divides $R\backslash Q$ into Q-equivalence classes. Let $W \subset (0, 1)$ be a set of representatives of the Q-equivalence classes. Now $R\backslash Q \subset \bigcup_{a,b \in Q}(aW + b)$ and $\lambda(aW + b) = a\lambda W$. It follows that $0 < \lambda W \leqq 1$.

We use the Vitali covering theorem to a.e. cover W with countably many pairwise disjoint closed intervals I_j with rational endpoints such that $\lambda I_j < 2^{-1}\lambda W$ for each j and $\sum_j \lambda(I_j) < (1 + 2^{-1})\lambda W$. For some index j, $\lambda(I_j) < (1 + 2^{-1})\lambda(I_j \cap W)$. Let K_1 be this I_j. Then

$$\lambda(W\backslash K_1) \geqq \lambda W - \lambda K_1 \geqq \lambda W - 2^{-1}\lambda W > 0.$$

We use the Vitali covering theorem to a.e. cover $W\backslash K_1$ with countably many pairwise disjoint closed intervals J_j with rational endpoints, and disjoint from K_1, such that $\lambda J_j < 2^{-1}(\lambda(W\backslash K_1))$ for each j and $\sum_j \lambda(J_j) < (1 + 2^{-2})\lambda(W\backslash K_1)$. For some index j, $\lambda(J_j) < (1 + 2^{-2})\lambda(J_j \cap W)$. Let K_2 be this interval J_j. Then

$$\lambda(W\backslash K_1\backslash K_2) \geqq \lambda(W\backslash K_1) - \lambda K_2 > 2^{-1}\lambda(W\backslash K_1) > 0.$$

We use the Vitali covering theorem to a.e. cover $W\backslash K_1\backslash K_2$ with countably many pairwise disjoin closed intervals L_j with rational endpoints, and disjoint from $K_1 \cup K_2$, such that $\lambda L_j < 2^{-1}\lambda(W\backslash K_1\backslash K_2)$ for each j and

Received by the editors on March 10, 1977.