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# EXAMPLES OF FIXED POINT FREE MAPS FROM CELLS ONTO LARGER CELLS AND SPHERES

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ABSTRACT. Let $n \geqq 2$. Let $A$ and $B$ be closed $n$ dimensional balls in Euclidean space such that $A \subset B$ and $A \neq B$. Two types of fixed point free maps, $f$ and $g$, from $A$ onto $B$ are obtained—$f$ is space filling on $\partial A$, i.e., $f(\partial A) = B$, and $g$ preserves the boundaries of $A$ and $B$ i.e., $g^{-1}(\partial B) = \partial A$. A fixed point free map from a Hilbert cube onto a larger Hilbert cube is obtained which preserves their pseudo-boundaries. Two fixed point free maps with special properties from the bottom half of $S^n$ onto the $n$ sphere $S^n$, $n \geqq 2$, are obtained. Under the first one the preimage of the North Pole is the Equator. Under the second one the preimage of the South Pole is the Equator and, in addition, the second one is monotone. In relation to these last two examples, the following theorem is proved. If $K$ is a proper nonseparating subcontinuum of $S^2$ and if $f$ is a monotone mapping from $K$ onto $S^2$ such that $f[\mathrm{Bd}(K)] \not\subset K$, then $f$ has a fixed point. This theorem is compared with the Knaster-Kuratowski-Mazurkiewicz fixed point theorem.

1. **Introduction.** Any continuous function from an arc onto a larger arc has a fixed point. This simple observation leads to the following question. Does every continuous function from a ball in a Euclidean space onto a larger ball have a fixed point? In various papers, conditions have been imposed on mappings $f$ from certain types of subcontinua of balls onto balls which imply that $f$ has a fixed point (for example, see [**1**], [**2**], [**4**], [**6**], [**9**], [**10**]). However, there seem to be no examples in the literature to show that the question above has a negative answer. In this paper we give such examples. In addition to being continuous and fixed point free, our mappings have certain special properties which lead to an example involving Hilbert cubes and some examples involving spheres. We then prove a fixed point theorem concerning the 2-sphere. This theorem is discussed in the light of our examples and a theorem in [**6**].

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