ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 11, Number 2, Spring 1981

SUBORDINATIONS FOR TYPICALLY-REAL AND RELATED FUNCTIONS

R. B. BYERS and E. P. MERKES

ABSTRACT. Let T be the typically-real functions in the open unit disk E and let C be the subclass of functions convex in the direction of the imaginary axis. For real μ, the subordination $z/2 + \mu a_2 z^2 \propto f(z) (z \in E)$ holds for all $f(z) = z + a_2 z^2 + \cdots$ in C if and only if $0 \leqq \mu \leqq 1/3$. If $\eta(z) = z/[1 + (1 - z^2)^{1/2}]$, then $2f(\eta(z))$ is in C whenever $f \in T$. From these results, we conclude for real μ that $z/4 + \mu a_2 z^2 \propto f(\eta(z)) \propto f(z) (z \in E)$ for all $f(z) = z + a_2 z^2 + \cdots$ in T if and only if $0 \leqq \mu \leqq 1/12$. If s_n is the n-th partial sum of $f(z) = z + a_2 z^2 + \cdots$, then subordinations of the form $s_n(\lambda z) \propto f(z) (z \in E)$, λ real, are obtained for $f \in T$ and $f \in C$.

1. **Introduction.** Let E denote the unit disk $|z| < 1$. The class C is all analytic functions $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$ in E such that $\{a_j\}$ is a real sequence and the intersection of $f(E)$ with each line parallel to the imaginary axis is empty or an interval. The functions $f \in C$ are univalent and the region $f(E)$ includes all points of the disk $|w| < \frac{1}{2}$ (see [**10**]). The convex univalent functions $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$ in E also map E onto a region that includes the $\frac{1}{2}$-disk [**8**, p. 45]. This property was extended to the subordinations

$$(1) \qquad \frac{1}{2} z \propto \frac{2}{3} z + \frac{1}{6} a_2 z^2 \propto f(z) = z + a_2 z^2 + \cdots (z \in E)$$

for all convex univalent functions f and it was shown that the constants 2/3 and 1/6 are best possible [**1**]. The subordination $2z/3 + a_2 z^2/6 \propto f(z) = z + a_2 z^2 + \cdots (z \in E)$ does not hold for all functions $f \in C$ since $f(z) = z/(1 - z^2)$ is in C and $f(E)$ omits the points ti for all real t, $|t| \geqq 1/2$. There is, nevertheless, an analog of (1) for the class C that is proved in this paper.

THEOREM 1. *Let μ be real. Then*

$$\frac{1}{2} z + \mu a_2 z^2 \propto f(z) = z + a_2 z^2 + \cdots (z \in E)$$

AMS(MOS) subject classification (1970). Primary 30A32 *Key words and phrases.* Typically real functions, subordination.

Received by the editors on June 18, 1979, and in revised form on October 26, 1979.