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INEQUALITIES FOR THE GENERALIZED TRANSFINITE DIAMETER

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ABSTRACT. Let E be a compact subset of a metric space X and f a Lipschitzian function on X. It is shown that $d(f(E)) \leqq Md(E)$, where d is the generalized transfinite diameter of Hille [2, 3] and M is the Lipschitz constant. Also, upper and lower bounds are obtained for the transfinite diameter of the union E of two "widely separated" compact sets, E_1 and E_2, in terms of the diameters of E, E_1, and E_2, the transfinite diameters of E_1 and E_2, and the distance between E_1 and E_2.

1. **Preliminaries.** The transfinite diameter in the complex plane was first introduced by Fekete in 1923. Pólya and Szegö extended the concept to three - dimensional space and showed that the transfinite diameter coincides with the capacity. Further generalizations of this concept were made by them and by Leja. Finally Hille, in two papers [2, 3], summarized and unified the previous generalizations. His papers contain bibliographies of previous work.

The present paper extends, in Theorem 1, a result of the author [6] from the complex plane to a metric space. Also inequalities are obtained for the transfinite diameter of the union of two "widely separated" sets.

2. **Averaging processes.** Let A be a function whose domain is all finite sequences of positive numbers. A is called an *averaging process* if it satisfies the following four axioms of Kolmogoroff [4]:

(i) $A[x_1, x_2, \ldots, x_n] > 0$ for all finite sequence $\{x_k\}$ of positive numbers;

(ii) A is a continuous, symmetric function of its arguments and is a strictly increasing function of each of them;

(iii) $A[x, x, \ldots, x] = x$; and

(iv) $A[x_1, x_2, \ldots, x_k, x_{k+1}, \ldots, x_n] = A[y, y, \ldots, y, x_{k+1}, \ldots, x_n]$ if $y = A[x_1, x_2, \ldots, x_k]$.

In addition, we will assume a fifth axiom:

(v) $A[kx_1, \ldots, kx_n] = kA[x_1, \ldots, x_n]$ for any $k > 0$.

We shall sometimes use the notation $A_{1 \leqq i \leqq n} x_i$ for $A[x_1, \ldots, x_n]$.

It can be proved [7] that an averaging process which satisfies (v) is of

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