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CLASS FIELD THEORY SUMMARIZED

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1. **Introduction.** A good starting point is a quote from M. J. Herbrand of which the following is a translation.

"There is perhaps no theory in science where at the same time the proofs are so difficult and the results of such perfect simplicity and of such great power." [**11**, p. 2].

Hopefully this summary will communicate the simplicity and power of the results of class field theory even though no proofs are presented—a fact which is bound to eliminate to some extent the sharp precision found in a complete course.

The first part of this summary will be a very classical presentation of class field theory such as it can be found in the work of Hasse, i.e., pre-World-War II class field theory. The second part will re-summarize class field theory in a more modern fashion using ideles, i.e., Chevalley's formulation.

2. **Goals.** What are the goals of class field theory? To answer this we need some definitions.

Let K be a finite extension of the rationals $\mathbf{Q}$. In fact, unless stated otherwise all fields discussed in this summary will be finite extensions of $\mathbf{Q}$. Let $\mathcal{O} = \mathcal{O}_K$ be the ring of algebraic integers of K. A *fractional ideal*, $\mathfrak{a}$, is a nonzero finitely generated $\mathcal{O}$-module where the generators are in K. So we can write $\mathfrak{a} = (\alpha_1, \ldots, \alpha_t)$ where the α's are the generators of $\mathfrak{a}$. If $\mathfrak{b} = (\beta_1, \ldots, \beta_s)$, we define the product $\mathfrak{a}\mathfrak{b} = (\ldots, \alpha_i\beta_j, \ldots)$ as the $\mathcal{O}$-module generated by the products of the various generators of $\mathfrak{a}$ and $\mathfrak{b}$. Under this multiplication the set of fractional ideals forms a multiplicative group, $A = A_K$, with $\mathcal{O} = (1)$ as the identity element.

Although nobody seems to want to define "the arithmetic of K", the following description seems to work in the present context. The *arithmetic of K* is the study of A, subgroups of A, factor groups of subgroups of A, groups isomorphic to these groups and certain ideals in A.

We can now state the three-fold goal of class field theory.

(I) "Describe" all finite abelian extensions of K in terms of the arithmetic of K. (L is an abelian extension of K if the Galois group $G(L/K)$ is abelian.)

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