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SOME GEOMETRIC PROPERTIES OF LORENTZ SEQUENCE SPACES

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Let $1 \leqq p < \infty$. For any $a = (a_1, a_2, \cdots) \in c_0 \backslash \ell_1, 1 = a_1 \geqq a_2 \geqq \cdots \geqq 0$, let

$$d(a, p) = \left\{ x = (\alpha_1, \alpha_2, \cdots) \in c_0 : \|x\| = \sup_{\sigma \in \pi} \left(\sum_{i=1}^{\infty} |\alpha_\sigma(\alpha_{\sigma(i)})|^p a_i \right)^{1/p} < \infty \right\}$$

where π is the set of all permutations of the natural numbers N. The Banach space $d(a, p)$ is called a Lorentz sequence space. The Lorentz sequence spaces in some sense are "weighted" ℓ_p-spaces. They possess some common properties with ℓ_p-spaces, but not always. For recent results on Lorentz sequence spaces, see [**1**, **2**, **3**, **4**, **5**].

It is known [**11**] that $d(a, p)$ is reflexive for every $a \in c_0 \backslash \ell_1$ when $1 < p < \infty$. However, in general, $d(a, p)$, $1 < p < \infty$, is not uniformly convex. In fact, it is known [**1**] that in $d(a, p)$, $1 < p < \infty$, uniform convexity, uniform convexifiability, and the condition $\inf_n s_{2n}/s_n > 1$ where $s_n = \sum_{i=1}^{n} a_i$, $n = 1, 2, \cdots$, are equivalent. In this paper, we show that if $1 < p < \infty$ then for every $a \in c_0 \backslash \ell_1$, $d(a, p)$ is locally uniformly convex.

A Banach space X is said to have the property (H) if X is strictly convex and for any sequence $\{x_n\}$ in X and x in X, $\lim_n \|x_n\| = \|x\|$ and $\{x_n\}$ converges weakly to x imply that $\lim_n \|x_n - x\| = 0$. The space X is said to have property (2R) if for any sequence $\{x_n\}$ in X such that $\|x_n\| = 1$, $n = 1, 2, \cdots$, if $\lim_{n,m} \|x_n + x_m\| = 2$ then $\{x_n\}$ is a Cauchy sequence in X. We show that every $d(a, p)$, $1 \leqq p < \infty$ has property (H) and if $1 < p < \infty$, then every $d(a, p)$ has property $(2R)$. Hence there exist Lorentz sequence spaces with property $(2R)$ but which are not uniformly convexifiable. It is known that Day's spaces [7] also possess these properties. We refer to [9, **10**] for the detailed study of properties (H) and $(2R)$.

A Banach space X is said to be locally uniformly smooth if for any x in X with $\|x\| = 1$ and for any $\epsilon > 0$, there exists a $\delta > 0$ such that $\|x + y\| + \|x - y\| \leqq 2 + \epsilon\|y\|$ for all y with $\|y\| \leqq \delta$. In § 2, we show that for all $a \in c_0 \backslash \ell_1$ and $1 < p < \infty$, $d(a, p)$ is locally uniform-

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