ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 7, Number 4, Fall 1977

PRIME IDEAL POSETS IN NOETHERIAN RINGS

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The prime ideals of a noetherian ring with the inclusion relation form a partially ordered set, which shall be called an N-poset. How can one tell if a given poset is an N-poset? Surprisingly little is known about this question. In [2, pp. 66–68], Hochster treats it briefly and offers a partial list of axioms dealing with the equivalent question of classifying the spectral topologies of noetherian rings. He also offers a question due to Kaplansky which provided the initial impetus for the work presented here. The original question is, "Must two primes, P, P' of height greater than one in a noetherian domain necessarily have a nonzero prime Q in their intersection?" Alternatively, it is equivalent to asking if the poset of nonzero primes in a noetherian domain can ever be decomposed into an (ordered) disjoint union of proper subsets. (Of course, we exclude the trivial case of height one maximal ideals.) More recently, it has been shown by McAdam [4], among others, that the answer is yes. Following this line of thought, we would like to know more about this decomposition. Primarily, what kind of component pieces can be used and how many (finite or infinite) of them can there be? To this particular aspect of the problem, this paper is addressed.

The crux of this paper is a technique which enables us to intersect certain collections of noetherian domains and obtain a new domain which is again noetherian. The poset of nonzero primes in this new domain decomposes into the disjoint union of the initial posets of nonzero primes. While the procedure is not completely arbitrary, the collection may be infinite and entirely new types of examples of N-posets can be formed. In § 2, we proceed to build some examples which seem particularly enlightening. Many more are possible. It is hoped that, in an area which has suffered from a paucity of examples, both the results and the construction leading to it will provide some relief.

NOTATION. K will be a fixed field. The symbols X, Y, Z unsubscripted or with one subscript will denote sets of indeterminates. A single indeterminate will always carry *two* subscripts. Cardinalities of sets will be denoted $|K|$, etc. The rings we shall employ in the construction will be localizations of polynomial rings; the letters f, g will be reserved for polynomials. $\mathfrak{D}$ will be an index set for a collection

Received by the editors on April 1, 1976, and in revised form on July 30, 1976.