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A WEAK HARTMAN'S THEOREM FOR HOMOMORPHISMS AND SEMI-GROUPS IN A BANACH SPACE*

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In this article we examine the extent to which Hartman's Theorem holds for homomorphisms and semi-groups in a Banach space. The technique used here for the main theorem is a modification of the technique of Moser's used by Pugh [4] to prove Hartman's Theorem for isomorphisms and groups in a Banach space.

Let E be a Banach space and let $L : E \rightarrow E$ be linear on E; possibly 0 is in the spectrum of L. A basic assumption throughout the paper is that L is hyperbolic; that is, $E = E^u \oplus E^s$ where $LE^u \subset E^u$ and $LE^s \subset E^s$, and $L^s \equiv L \mid E^s$ is a contraction while $L^u \equiv L \mid E^u$ is invertible and $(L^u)^{-1}$ is also a contraction. We let $k \equiv \max\{|L^s|, |(L^u)^{-1}|\} < 1$. It is not hard to prove that if the spectrum of L has no points on the unit circle, then L is hyperbolic in some norm on E. Assume that E is given the norm $|x + y| = \max\{|x|, |y|\}$ for $x \in E^u$, $y \in E^s$.

Let $\beta(a)$ denote the set of bounded maps $\lambda : E \rightarrow E$ such that $|\lambda(x) - \lambda(y)| \leqq a|x - y|$ and $\lambda(0) = 0$. We use $\Lambda = L + \lambda$ and $\Lambda' = L + \lambda'$ for $\lambda, \lambda' \in \beta(a)$. We use 1 to denote an identity map.

We now state Pugh's version of Hartman's Theorem for isomorphisms for reference purposes:

THEOREM 1. *If L is an isomorphism and a is small enough, then for each Λ there is a unique bounded, uniformly continuous map $g : E \rightarrow E$ such that if $h = 1 + g$, then*

$$hL = \Lambda h. \tag{1}$$

Furthermore h is a homeomorphism depending continuously on λ.

Equation (1) implies that h maps orbits of L into orbits of Λ and vice versa.

Hale gives the example [1]

$$\dot{x}(t) = 2\alpha x(t) + N(x_t) \tag{2}$$

where $\alpha > 0$, $N(0) = 0$, and the Lipschitz constant of N in the ϵ-ball at 0 goes to 0 as $\epsilon \rightarrow 0$. Considered as a delay equation, (2) generates a strongly continuous semi-group $T(t)$ defined on $C([-r, 0], \mathbf{R}^n)$. If $N = 0$, the range of $T(r)$ is one dimensional. It is not hard to con-

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