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EXCHANGEABLE EVENTS AND COMPLETELY MONOTONIC SEQUENCES

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1. **Introduction.** We investigate the probability theory of an infinite sequence of events, all having the same probability, and we assume the events are exchangeable. That is, for each positive integer n, each pair of n-fold intersections of events have the same probability. In the familiar game of tossing a coin forever, the events "heads" provide an example of a sequence of exchangeable events. A less trivial example is provided by Pólya's urn model. (See, for example, Feller [**1**, p. 226].)

We show in Theorem 1 that a sequence of events of common probability is exchangeable if and only if the sequence of real numbers whose nth term is the probability common to the n-fold intersections of events is a completely monotonic sequence. Theorem 2 asserts that for such events, Kolmogorov's Strong Law of Large Numbers holds if and only if the events are independent. Theorems 3, 4, 5, and 6 describe probabilities of unions and intersections of exchangeable events of common probability.

We recall now preliminaries from Feller [**1**, p. 225], Widder [**4**, p. 108 and p. 12], and Hardy [**2**, pp. 279–282]. For a sequence $\mu_0, \mu_1, \cdots$ of real numbers, we denote by $\Delta^m \mu_q$ the sum

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} \mu_{q+m-k}$$

and define $\mu_0, \mu_1, \cdots$ to be completely monotonic if

$$(-1)^m \Delta^m \mu_q \geqq 0 \quad \text{for } m, q = 0, 1, \cdots,$$

and minimally completely monotonic if for every $\mu < \mu_0$ the sequence $\mu, \mu_1, \mu_2, \cdots$ is not completely monotonic.

A fundamental theorem of Hausdorff is that $\mu_0, \mu_1, \cdots$ is completely monotonic if and only if there exists a bounded nondecreasing function α from $[0, 1]$ into $[0, \infty)$ such that

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