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ON AN INTEGRAL INEQUALITY FOR DIVERGENCE-FREE FUNCTIONS

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1. **Introduction.** Let D be a bounded, two-dimensional domain with smooth boundary ∂D and $\phi_i(x_1, x_2) = \phi_i(x)$ $[i = 1, 2]$ any sufficiently smooth vector-valued function which is defined on D, vanishes on ∂D, and satisfies the divergence-free condition, $\phi_{j,j} = 0$ in D. Here the summation convention is used and a comma denotes differentiation; for example,

$$\phi_{j,j} = \frac{\partial \phi_1}{\partial x_1} + \frac{\partial \phi_2}{\partial x_2} .$$

Of interest in this work is the calculation of a positive constant λ such that

$$\int_D \phi_i \phi_i \, dx \leqq \frac{1}{\lambda} \int_D \phi_{i,j} \phi_{i,j} \, dx \tag{1}$$

when D can be enclosed in a wedge of angle π/α, $\alpha > \frac{1}{2}$. Ideally we would like to calculate an optimal value for λ; however, this does not seem possible and we shall, therefore, sharpen known results.

Inequality (1) has been employed in stability and uniqueness studies for the Navier-Stokes equations (see e.g. Serrin [11]) and in an examination of growth properties of solutions for a model of a dusty gas system (see Crooke [2]), among other applications.

It is generally possible to establish these types of inequalities by considering a corresponding variational problem. For inequality (1) we are interested in the following variational problem:

$$\hat{\lambda} = \inf_{\psi_i \in \Gamma(D)} \frac{\int_D \psi_{i,j} \psi_{i,j} \, dx}{\int_D \psi_i \psi_i \, dx} \tag{2}$$

where $\Gamma(D)$ denotes the class of Dirichlet integrable, vector-valued functions which are defined on D, vanish on ∂D and satisfy $\psi_{j,j} = 0$ in D. Hence, if λ is any lower bound for $\hat{\lambda}$ and ϕ_i any function belonging to $\Gamma(D)$, then

$$\lambda \leqq \inf_{\psi_i \in \Gamma(D)} \frac{\int_D \psi_{i,j} \psi_{i,j} \, dx}{\int_D \psi_i \psi_i \, dx} \leqq \frac{\int_D \phi_{i,j} \phi_{i,j} \, dx}{\int_D \phi_i \phi_i \, dx} ,$$

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