ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 3, Number 3, Summer 1973

GENERAL RADICAL THEORY IN RINGS

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1. **The general radical theory.** A radical property, speaking roughly, is one which can be "divided out". Thus, for example, any abelian group G has a largest torsion subgroup $H = \{x \in G \mid nx = 0$ for some positive integer $n\}$ such that G/H is torsion-free. Another perfectly typical example is the sum I of all nil ideals (every element nilpotent) of a ring R. It is easy to see that I is itself a nil ideal and that R/I has no nonzero nil ideals. It will be clear as we go along that most of what we do could just as well be done in a much more general category (yielding, as special cases, the parallel theories in groups, rings, modules, algebras, and so on). However, to avoid too much generality we will stick to rings and agree that (unless otherwise stated) all rings considered will belong to some arbitrary (but fixed) universal class W of not necessarily associative rings. Mostly we can as well take W to be the class of all such rings, but the class could be more restricted, provided that it has the properties:

(1) Hereditary; that is, $I \lhd R \in W$ implies $I \in W$ (where $I \lhd R$ means I is an ideal of R).

(2) Homomorphically closed; that is $R \in W$ implies all $R/I \in W$.

One of the motivations for studying radicals is that often the "dividing out" process yields a ring which in some sense is simpler than the original one and hence possibly more amenable. A typical example is the Wedderburn-Artin theorem: If R is a (associative) ring with descending chain condition on left (or right) ideals and I is the sum of all nilpotent ideals of R, then R/I is the direct sum of a finite number of simple rings each a complete matrix ring over some division ring. Even more, it is always hoped that one may be able to extract information about the original ring. For example, if the Jacobson radical J of a ring R is nil then any idempotent element u (that is $u^2 = u$) of R/J has an inverse image in R which is idempotent. Much of what can be done will of course depend on the kind of ring and the particular radical. However quite a bit can be said about radicality in general, and in any case it is worthwhile looking at the general theory, if only to bring a certain amount of order into the bewildering tangle of results in the area.

Received by the editors June 30, 1971 and, in revised form, October 10, 1971.
AMS (MOS) subject classifications (1970). Primary 16A21, 16–02, 17–02; Secondary 18E40.