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# MORSE THEORY IN HILBERT SPACE

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To the Memory of Heinz Hopf

1. **Introduction.** Let $E$ be a real Hilbert space, and let $f$ be a real valued function whose domain $V$ is a subset of $E$ to be specified later. Morse theory deals with the critical points of $f$, i.e., those points $x_0 \in V$ at which the Fréchet differential $df(x; h)$ (see e.g. [2]) of $f$ is zero (identically in $h$). Since, for any $x \in V$, the differential $df(x; h)$ is a linear bounded functional in $h$, there exists a unique element $g = g(x)$ in $E$, called the gradient of $f$, such that

$$df(x; h) = \langle g(x), h \rangle, \tag{1.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $E$. (In a finite-dimensional space this definition is easily seen to agree with the usual one of $\operatorname{grad} f$ as the vector whose components are the partial derivatives of $f$.) It follows that a point $x \in V$ is critical if and only if it satisfies the equation

$$g(x) = 0. \tag{1.2}$$

Related to the notion of a critical point is that of a critical level:

DEFINITION 1.1. A critical level (or value) of $f$ is a real number $c$ such that $f(x) = c$ for at least one critical point $x$.

Two problems arise naturally:

PROBLEM I. Describe the "nature" of a critical point, a local problem.

PROBLEM II. Obtain an estimate of the number of critical points in terms of geometrical (topological) properties of the domain $V$ of $f$, a problem in the large.

Problems I and II will be discussed in §§2 and 3 respectively in detail. At the present moment we confine ourselves to some introductory and intuitive remarks concerning these problems in very simple cases. These remarks motivate the use of the more abstract notions employed later, in particular the use of the singular homology theory. A review of the relevant definitions and facts of this theory forms the last part of this introduction.

*A simple case of Problem* I. Let $E = E^2$ be the Euclidean plane of points $x = (x_1, x_2)$, let $V = D$ be a disc with center $\theta = (0, 0)$, and let $f$ be a diagonalized quadratic form. We consider the cases

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