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SOME CHARACTERIZATIONS OF FIRST NEIGHBORHOOD COMPLETE IDEALS IN DIMENSION TWO

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ABSTRACT. In this paper we study first neighborhood complete ideals in a two-dimensional normal Noetherian local domain $(R, \mathfrak{M})$ with algebraically closed residue field and the associated graded ring an integrally closed domain. It is shown that a complete quasi-one-fibered $\mathfrak{M}$-primary ideal I of R is a first neighborhood complete ideal if and only if $e(I) = e(\mathfrak{M}) + 1$. This implies that $(R, \mathfrak{M})$ is a rational singularity if and only if a (every) first neighborhood complete ideal has minimal multiplicity. Moreover, if $(R, \mathfrak{M})$ has minimal multiplicity, then a complete quasi-one-fibered $\mathfrak{M}$-primary ideal I of order one is a first neighborhood complete ideal if and only if certain numerical data associated with I are minimal. This yields a simple proof of the fact that first neighborhood complete ideals in such a local ring R are projectively full.

1. Introduction. Let $(R, \mathfrak{M})$ be a two-dimensional Muhly local domain, that is, a two-dimensional normal Noetherian local domain with algebraically closed residue field and its associated graded ring an integrally closed domain. It follows that the $\mathfrak{M}$-adic order function ord_R is a valuation (denoted by $v_{\mathfrak{M}}$), and the blowup $\text{Bl}_{\mathfrak{M}} R$ of R at $\mathfrak{M}$ is a desingularization of R.

In [**2**] complete $\mathfrak{M}$-primary ideals I adjacent from below to $\mathfrak{M}$ (that is, length$(\mathfrak{M}/I)$ equals 1) have been studied. It has been proved in [**2**, Theorem 3.2, page 1144] that these ideals are precisely the inverse transforms in R of the maximal ideals $\mathfrak{M}'$ of the immediate quadratic transforms $(R', \mathfrak{M}')$ of $(R, \mathfrak{M})$. Therefore, these ideals have been called *first neighborhood complete ideals.*

In [**2**, Lemma 3.1, page 1144], it has been shown that, if the first neighborhood complete ideal I is the inverse transform of $\mathfrak{M}'$, then the set $T(I)$ of Rees valuations satisfies $T(I) \subseteq \{v_{\mathfrak{M}}, w\}$ and $w \in T(I)$, where w denotes the $\text{ord}_{R'}$-valuation (w is called an immediate prime

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