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On a Frobenius reciprocity theorem for locally compact groups

Dedicated to Professor Hisaaki YOSHIZAWA on his 60th birthday

By

Hitoshi SHIN'YA

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§0. Introduction

Let G be a locally compact unimodular group, and S a closed subgroup of G. Suppose that there exists a compact subgroup K of G with $G=SK$ ($S \cap K$ is not necessarily trivial). This paper is devoted to give a generalization of the Frobenius reciprocity theorem for these G and S.

In particular, let G be a finite group, and S a subgroup of G. Let $\{\mathfrak{H}, T(x)\}$, $\{H, \Lambda(s)\}$ be representations of G, S on finite-dimensional vector spaces $\mathfrak{H}$, H respectively. We shall denote by $\{\mathfrak{H}^{\Lambda}, T^{\Lambda}(x)\}$ the representation of G induced from $\{H, \Lambda(s)\}$, and by $\{\mathfrak{H}, T_S(s)\}$ the restriction of $\{\mathfrak{H}, T(x)\}$ to S. Then the Frobenius reciprocity theorem can be stated in the following three forms which are mutually equivalent.

(1) If $\{\mathfrak{H}, T(x)\}$ and $\{H, \Lambda(s)\}$ are irreducible, then $\{\mathfrak{H}^{\Lambda}, T^{\Lambda}(x)\}$ contains $\{\mathfrak{H}, T(x)\}$ exactly as many times as $\{\mathfrak{H}, T_S(s)\}$ contains $\{H, \Lambda(s)\}$.

(2) $\mathrm{Hom}_S(H, \mathfrak{H}) \cong \mathrm{Hom}_G(\mathfrak{H}^{\Lambda}, \mathfrak{H})$ (linearly isomorphic).

(3) $\mathrm{Hom}_S(\mathfrak{H}, H) \cong \mathrm{Hom}_G(\mathfrak{H}, \mathfrak{H}^{\Lambda})$.

Various generalizations of this theorem were given by many people. Using the direct integral decomposition theory, both F. I. Mautner and G. W. Mackey generalized the theorem as stated in form (1) above. In Mautner's case, G is assumed to be a separable locally compact unimodular group and S a compact subgroup [7]. In Mackey's case, G is a separable locally compact group and S a closed subgroup of G [6]. But, in his case, the Frobenius reciprocity theorem is formulated only for representations which appear in the direct integral decompositions of regular representations. R. Penney also formulated in [9] a generalization of the Frobenius reciprocity theorem in form (1). He dealt with Lie groups and made use of the C^{∞}-vector method.

In the case of C. C. Moore [8], the group G is a locally compact group and S a closed subgroup of G. He assumed that the homogenous space $S\backslash G$ possesses an invariant measure and that both $\{\mathfrak{H}, T(x)\}$, $\{H, \Lambda(s)\}$ are unitary. Nevertheless the induced representation $\{\mathfrak{H}^{\Lambda}, T^{\Lambda}(x)\}$ is defined so as to be an isometric one on a Banach