J. Math. Kyoto Univ. (JMKYAZ)
26-1 (1986) 1-11

On the time evolution of the Boltzmann entropy

By

Shunji ITO*, Masahiro MIZUTANI** and Toshio NIWA*

1. Introduction.

Among the fundamental problems of statistical mechanics one of the utmost interest is the justification of the second law of thermodynamics, or the explanation of the apparent conflict between the microscopic dynamical reversibility and the macroscopic irreversibility. The following things are well known; 1) the entropy (fine-grained entropy) does not change with time and the values of dynamical variables invariant with respect to the time reversal transformation ($t \to -t$, $p \to -p$) can not increase monotonically for all initial conditions because of the reversibility of mechanics, and 2) any continuous dynamical variable can not evolve monotonically all along the time because of Poincarè's recurrence theorem. Several ways to avoid these difficulties are to take the definitions of entropy based not on the fine-grained description provided by an ensemble density, which satisfies Liouville's equation, but on the coarse-grained [1] [2] [3] [4] or reduced description associated with a kinetic equation such as Boltzmann's equation [5] [6] or the master equation [7]. These definitions depend on the averaging of the ensemble density on the phase-space cell in the case of the coarse-grained description and on the assumption under which the kinetic equation holds or on the introduction of Markov process in the case of the reduced description. A quite different definition is proposed by Prigogine [8] and Misra [9]. In their definition the entropy does not appear to have the additive property [10].

For the isolated finite dynamical system it is impossible, as already we noted, to define rigorously a nonequilibrium entropy as a dynamical variable if we require that the increasing law of the entropy should be strictly realized. But these circumstance will not deny the possibility to be able to prove the increasing law of the entropy rigorously in the following sense; there exists some increasing function $f(t)$ and some small $\varepsilon > 0$ for almost nonequilibrium initial states ω such that the inequality $|S(T_t\omega) - f(t)| < \varepsilon$ holds for a very long time interval, where $T_t\omega$ represents the time evolutuin of ω. Of course it is an extremely difficult problem to prove this statement exactly.

In this paper we prove the above statement in a somewhat weakened form, namely we introduce the "Boltzmann entropy $S(\omega)$", which will be defined, and prove the inequality $S(T_t\omega) \geqq S(\omega) - \varepsilon$ for almost ω and a very long time interval. This

Communicated by Professor H. Yoshizawa, March 12, 1984.